Evolution of Conventions in Endogenous Social Networks*

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Abstract

We analyze the dynamic implications of learning in a large population coordination game where both the actions of the players and the communication network evolve over time. Cost considerations of social interaction are incorporated by considering a circular model with endogenous neighborhoods, meaning that the locations of the players are fixed but players can create their own communication network.

The dynamic process describing medium-run behavior is shown to converge to an absorbing state, which may be characterized by coexistence of conventions. In the long run, when mistake probabilities are small but nonvanishing, coexistence of conventions is no longer sustainable as the risk-dominant convention becomes the unique stochastically stable state.

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1 Introduction

In a wide variety of economic and social environments, an agent's utility depends on successful coordination with other individuals. The following two examples illustrate this point. First, as suggested by Lewis [19], suppose that oligopolists are confronted with a change in the price of their raw material and therefore must set new prices of their product. It is to no one's advantage to set his price higher than the others set theirs, since if he does, he tends to lose his share of the market. Nor is it to anyone's advantage to set his price lower than the others set theirs, since if he does, he menaces his competitors and incurs their retaliation. Hence, each competitor must set his price close to the price he expects the others to set. Second, as described by Diamond [7], in various parts of the world in the early stages of food production hunter-gatherer societies were confronted with the introduction of cultivation of plants and the domestication of animals. It was to one's advantage to coordinate in either hunting and gathering or food production.[1] Once coordination has been achieved on a certain behavior, then it is likely that this behavior will become the convention. For this reason Lewis [19] and Schelling [22] already stated that a convention should be considered a solution to a coordination problem. More precisely, Lewis defines a convention as a behavioral regularity such that everyone conforms to the regularity, expects others to conform, and wants to conform given that others conform.

The above examples illustrate two fundamental factors important in determining optimal behavior when agents face a coordination problem. An agent's expectation about the behavior of others plays a significant role. But underlying those expectations is an interaction structure governing communication between the players. Implicit in our discussion of conventions, we find ourselves talking about localities: geographic or social. Diamond [7] stresses the local, spatial interaction throughout his thesis.[2] Other examples of well-known conventions are languages, currencies, product standards, codes of dress and accounting standards.

[1] Although, as Diamond points out on page 105 that most peasant farmers and herders weren't necessarily better off than hunter-gatherers: "Archeologists have demonstrated that the first farmers in many areas were smaller and less well-nourished, suffered from more serious diseases, and died on the average at a younger age than the hunter-gatherers they replaced." The explanation offered for the increase in farming and herding communities is that individuals were seeking to minimize the risk of starvation.

[2] On page 103 Diamond writes,"In short, only a few areas of the world developed food production independently, and they did so at wildly differing times. From those nuclear areas, hunter-gatherers of some neighboring areas learned food production,..."

Overview of the Model

We analyze the dynamic implications of learning in a large population coordination game where agents are distributed spatially, and both the actions of the players and the communication network between these players evolve over time. We follow the conventional evolutionary game theoretic models on coordination problems in assuming that players use the same pure strategy against all opponents they interact with, i.e., the players with whom they communicate, and we allow for this strategy to be adjusted over time. We depart from the conventional models in assuming that the interaction network itself is also subject to evolutionary pressure. Jackson and Watts [15] develop a similar setting; we depart from that model by incorporating cost considerations of social interaction. Instead we devise a circular model with an endogenous communication network, meaning that the locations of the players are fixed but players can create their own interaction neighborhood by forming and severing links with other players. We assume that the larger the distance between two players on the circle, the larger the maintenance costs of the mutual link will be. As maintenance costs include invested time and effort, distance should not only be interpreted as physical distance but may also represent social distance.

Players typically react myopically to their environment by deciding about both pure strategies and links based on a best-reply dynamics. Sometimes, however, players make mistakes when implementing their decisions, or alternatively players experiment with nonoptimal replies. Whether or not these mistakes should be included explicitly in the model depends on the span of time over which we are interested in the players' behavior as predicted by the model. As explained by Binmore, Samuelson and Vaughan [5] the model corresponds to the players' medium-run behavior in the absence of the perturbations representing the players' mistakes. We find that in this case, the dynamic process converges to an absorbing state. As the set of absorbing states includes states in which different kinds of behavior are observed, the population's medium-run behavior is possibly characterized by coexistence of conventions. In the long run, i.e., when the perturbations representing the players' mistakes are taken into account, coexistence of conventions is no longer possible. Namely, the risk-dominant convention is the unique stochastically stable convention, meaning that it will be observed almost surely when the mistake probabilities are small but nonvanishing.

Related Literature

As stated earlier, Jackson and Watts study an endogenous model of network formation. Although our models differ in cost considerations we obtain similar insights. Theorem 4 in Section 4 is equivalent to their first main result within a spatial setting.

Related to the present paper is also the literature on network formation. Dynamic models of network formation are considered by e.g. Bala and Goyal [2], Jackson and Watts [14], and Watts [23]. In these models the presence of a link however does not indicate that the involved players interact by means of playing a game. Instead there is a deterministic benefit from an emerging network. The best-known and most intuitive example is the symmetric connections model as introduced by Jackson and Wolinsky [16], which represents social communication between agents. In fact, agents communicate with all other agents they are directly or indirectly connected with. The value of the communication depends on the number of links involved in the shortest path that connects a pair of agents. Watts [23] shows convergence to an efficient network in case costs are small and closer connections are valued more than distant connections. Note that the model is deterministic and therefore involves initial state dependence. Bala and Goyal [2] also find convergence to efficient networks in their deterministic setting. Their model differs from Watts [23] as it focuses on directed networks and players receive the same benefit from direct and indirect connections. Jackson and Watts [14] consider a stochastic model similar to Kandori, Mailath and Rob [17] and Young [24]. They find that the networks that occur with positive probability in the stationary distribution when mistake probabilities go to zero are either stable or a cycle. In their model efficiency can not be ensured.

The evolution and stability of conventions is analyzed through the population adjustment models with persistent randomness in Kandori, Mailath and Rob [17], Young [24], and Ellison [8]. With respect to coordination games these learning models identify the risk-dominant equilibrium as the unique long-run convention. Goyal and Janssen [12] focus on nonexclusive conventions in a deterministic framework to model the idea that, at the expense of some additional costs, players can remain flexible and therefore coordinate their actions more successfully. They find that the Pareto-efficient or risk-dominant equilibrium prevails depending on whether these costs are low or high, respectively. Furthermore, at intermediate cost levels, the two conventions coexist. Coexistence of exclusive conventions is also a feature of the model with noise on the margin analyzed by Anderlini and Ianni [1].

The models mentioned above deal with exogenously given patterns of interaction. In particular, Kandori, Mailath and Rob [17] and Young [24] use uniform matching rules, meaning that every player possibly interacts with all other players, while Anderlini and Ianni [1], Ellison [8], and Goyal and Janssen [12] use local matching rules, expressing that every player can only interact with a small subset of the population. However, by fixing the pattern of interaction exogenously these models ignore that players may have the desire and, at least to some extent, the ability to affect the set of players with whom they interact.

Endogenous patterns of interaction in population adjustment models can also be realized by allowing players to migrate, i.e., choose their location. Bhaskar and Vega-Redondo [4], Ely [9], Mailath, Samuelson and Shaked [20], and Oechssler [21] show that migration implies, or at least allows for, the population to coordinate on the Pareto-efficient equilibrium. In particular, Oechssler [21] shows that in his deterministic framework, given that all conventions are initially present, the efficient one will eventually prevail throughout. Ely [9] considers a stochastic model and shows by considering the stationary distribution when mistakes probabilities become small that the efficient convention will occur independent of the initial conditions. In both models coexistence of conventions is not possible. Bhaskar and Vega-Redondo [4], in a model with asynchronous strategy and location revision opportunities, indicate the possibility of coexistence of conventions in the medium run when the game is a 'pure' coordination game. Also in the long run both conventions are possible, i.e., both appear with positive probability in the stationary distribution of the stochastic model, in the case of frequent play. Otherwise, only the efficient convention is stochastically stable. Finally, Mailath, Samuelson and Shaked [20] look at a quite different context in continuous time. Players of two continuum populations have to decide which location to visit. They show that if the evolutionary process is monotonic and players can avoid undesirable matching, then every locally stable configuration is efficient.

The paper is organized as follows. In Section 2 we introduce the model. Section 3 deals with the deterministic model describing the medium-run behavior of the dynamic process. Long-run behavior is analyzed in Section 4 by means of the stationary distribution of the stochastic model when mistakes are small but nonvanishing. The robustness of the obtained results with respect to the exact specification of the model is discussed in Section 5. Finally, Section 6 concludes and Section 7 contains the proofs.

2 The Model

Consider a large but finite population of players $N = \{1, ..., n\}$ who are spatially distributed around a circle. Players are distributed around the circle in an equidistant fashion. Each discrete-time period $k \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ the players' interaction consists of two stages. In the first stage players form or sever links connecting themselves to other players. In the second stage pairs of players who are linked play a coordination game and adjust their actions.

The state s_k of the dynamic process in each period k is given by a graph g_k, with its nodes representing the players and its edges capturing the established communication links, and an action profile a_k, specifying the action being played by every player $i \in N$. Let $\mathcal{G}$ and $\mathcal{A}$ denote the set of possible graphs and the set of possible action profiles, respectively, then $s_k = (g_k, a_k) \in \mathcal{G} \times \mathcal{A} = \mathcal{S}$.

Each period, the presence of a link between players i and j, with $i, j \in N$ and $i \neq j$, results in a maintenance cost c_{ij} to both players. The costs c_{ij} are determined by the distance between the two players on the circle. Formally, we assume that

$$c_{ij} = \min \{\gamma |j - i|, \gamma |j - i + n|\},$$

where $\gamma \geqq 0$ are the so-called unit costs.

Let $\mathcal{L}_{i,k} \subset N$ denote the set of players that player $i \in N$ is linked with after the first stage in period $k \in \mathbb{N}_0$. We refer to $\mathcal{L}_{i,k}$ as the *interaction neighborhood* of player i at time k. In addition, with slight abuse of notation, we write $a_k = \left(a_{i,k}, a_{\mathcal{L}_{i,k},k}, a_{-\mathcal{L}_{i,k},k}\right)$. Note that when not causing any confusion $\mathcal{L}_{i,k}$ and $a_{i,k}$ may also be denoted by $\mathcal{L}_i$ and a_i, respectively. In the second stage of each period k, a player i plays a coordination game with all players $j \in \mathcal{L}_{i,k}$. The gross benefits are determined by the utilities u in the 2×2 coordination game given below.

	A	**B**
A	a, a	c, d
B	d, c	b, b

(1)

The payoff to player i in period k is given by

$$\pi_i \left(a_{i,k}, a_{\mathcal{L}_{i,k},k}\right) = \begin{cases} \sum_{j \in \mathcal{L}_{i,k}} u\left(a_{i,k}, a_{j,k}\right) & \text{if } \mathcal{L}_{i,k} \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_{i,k} \in \{A, B\}$, for all $i \in N$ and $\mathbb{N}_0$. The net benefits of a player i can be found by subtracting all maintenance costs, $\sum_{j \in \mathcal{L}_{i,k}} c_{ij}$, from his gross benefits. For notational convenience only we assume that $a, b, c, d \in \mathbb{N} \cup \{0\}$. Furthermore, it is required that $a > d \geqq 0$ and $b > c \geqq 0$, implying that both (A, A) and (B, B) are strict Nash equilibria. Furthermore, the payoffs a and b are taken to be larger than or equal to γ for the model to be nontrivial, i.e., for the players to have an incentive to interact. Finally, we assume that $a + c > b + d$, implying that (A, A) is the risk-dominant equilibrium as defined by Harsanyi and Selten [13]. Note that when the two actions have equal security levels, i.e., $c = d$, (A, A) is also the Pareto-efficient equilibrium.

Now, we describe how players establish and sever links, and how they adjust their pure strategies. The process of link formation and link severance is based on the process described in Jackson and Watts [14] and similar to Jackson and Watts [15]. In the first stage of each period k two players i and j, $j \neq i$, are randomly selected with probability $\rho_{ij} > 0$. Hence, we consider a matching process with players randomly meeting each other in pairs, and time periods being identified with the encounters. Only the two players constituting that pair can alter their (potential) mutual link in period k. If their mutual link already exists, they decide whether to sever it, and otherwise they decide whether to create the link.

The part of the dynamic process that describes how the two selected players establish or sever a mutual link is modelled as follows. First, suppose player i and j are not linked and therefore have to decide whether to establish the link ij. Player i compares the payoff he would have received from playing the coordination game with player j in the previous period with the maintenance costs c_{ij} of the link, i.e.,

$$u\left(a_{i,k-1}, a_{j,k-1}\right) \geqq c_{ij}.$$

Of course, player j follows exactly the same procedure. Only in case both payoffs are greater than or equal to the costs c_{ij}, the players decide to establish the mutual link. Notice that according to the deterministic part of the link formation process, two players will never establish a link if their mutual distance on the circle exceeds $\max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Namely, in that case the maintenance costs are always larger than the payoff obtained in the coordination game. Second, suppose that at the moment players i and j are linked, and that they therefore have to decide whether to sever their mutual link. Either player can sever the link ij unilaterally if $u\left(a_{i,k-1}, a_{j,k-1}\right) \leqq c_{ij}$.

We assume that every linking decision made by a pair of players is implemented correctly with probability $1-2\tau$, where $\tau \geqq 0$. With probability 2τ, the two potential linking decisions are implemented with equal probability.

To describe the adjustment of the players' pure strategies, we follow Ellison [8] by assuming that players typically react myopically to their environment. In fact, we assume that in the second stage of period t, player i chooses

$$a_{i,k} \in \arg\max_{a_i} \pi_i \left(a_i, a_{\mathcal{L}_{i,k},k-1}\right) \tag{2}$$

with probability $1-2\varepsilon$, where $\varepsilon \geqq 0$. With probability 2ε player i chooses between the two pure strategies at random with equal probability. Player j adjusts his pure strategy in the same way.

We refer to Section 5 for a discussion of alternative specifications of the dynamic process.

For a given initial state $s_0 \in \mathcal{S}$, the dynamic process introduced above defines a Markov process $\{s_k\}_{k\in\mathbb{N}_0}$ in discrete time with the finite state space $\mathcal{S} = \mathcal{G} \times \mathcal{A}$. Let $P := P(\tau, \varepsilon)$ denote the (one-step) transition matrix of the Markov process. Then, if $s = (g, a)$, $\tilde{s} = (\tilde{g}, \tilde{a}) \in \mathcal{S}$, the entry $P_{s\tilde{s}} := P_{s\tilde{s}}(\tau, \varepsilon)$ of the transition matrix is the probability that the state is $\tilde{s}$ at time $k+1$ conditional on the state being s at time k, i.e.,

$$P_{s\tilde{s}} = \Pr\left(s_{k+1} = \tilde{s} \mid s_k = s\right).$$

Let $P^l := P^l(\tau, \varepsilon)$ denote the l-step transition matrix of the Markov process, then the entry $P^l_{s\tilde{s}} := P^l_{s\tilde{s}}(\tau, \varepsilon)$ of the l-step transition matrix is equal to

$$P^l_{s\tilde{s}} = \Pr\left(s_{k+l} = \tilde{s} \mid s_k = s\right).$$

We refer to the Markov process $\{s_k\}_{k\in\mathbb{N}_0}$ as *adaptive play.*

3 Adaptive Play without Mistakes

This section deals with the model introduced in section 2 when both mistake probabilities, τ and ε, are equal to zero. We refer to this specific Markov process as *adaptive play without mistakes.* As discussed extensively by Binmore, Samuelson and Vaughan [5] adaptive behavior without mistakes describes the medium-run behavior of the dynamic process, where the medium-run refers to the time span needed for the dynamic process to reach the neighborhood of the first equilibrium near to which it will stay for a significant period of time.

Analyzing the dynamic process in the medium run boils down to identifying the absorbing states of the Markov process representing adaptive play without mistakes. An absorbing state is a state that once entered cannot be left. With respect to the present model this implies that in an absorbing state no single player wants to sever any of the links he is involved in, no pair of players wants to establish a mutual link or adapt his pure strategy. Below, we discuss what the absorbing states of adaptive play without mistakes look like, and we explore conditions which guarantee convergence to one of these absorbing states.

A convention is defined by Lewis [19] as a pattern of behavior that is customary, expected, and self-enforcing, meaning that everyone conforms, expects others to conform, and wants to conform given that everyone else conforms, respectively. Since a convention is self-enforcing, it has to be an absorbing state of the Markov process describing adaptive play without mistakes. It can easily be verified that the present model has two absorbing states which are conventions. Namely, the state such that

$$\begin{cases} a_i = A, \\ j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{a}{\gamma}, \end{cases} \tag{3}$$

for all $i \in N$ and the state such that

$$\begin{cases} a_i = B, \\ j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{b}{\gamma}, \end{cases} \tag{4}$$

for all $i \in N$. We refer to (3) as the $\left(A, \frac{a}{\gamma}\right)$-convention or the risk-dominant convention and to (4) as the $\left(B, \frac{b}{\gamma}\right)$-convention.

The $\left(A, \frac{a}{\gamma}\right)$-convention and $\left(B, \frac{b}{\gamma}\right)$-convention are, however, not the only absorbing states of adaptive behavior without mistakes. The other absorbing states indicate that the players' medium-run behavior may be characterized by coexistence of conventions. The following example shows existence of an absorbing state in which players act differently. This absorbing state therefore indicates the possibility of coexistence of conventions in the medium-run, see also Bhaskar and Vega-Redondo [4] for a similar observation.

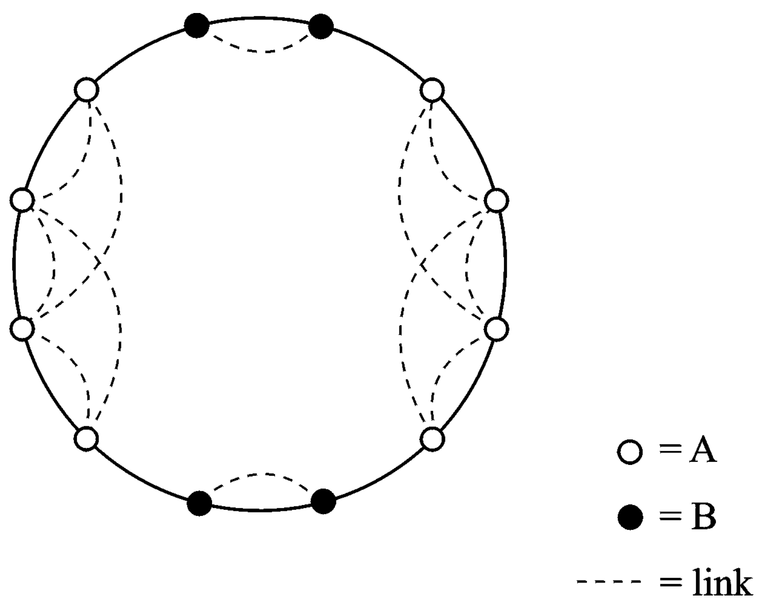

Figure 1: An absorbing state reflecting co-existence of conventions.

Example 1 Consider a population consisting of 12 players. Assume $\gamma = 1$ and let the payoffs of the coordination game be given by $a = 2$, $b = 1$, and $c = d = 0$. It can easily be verified that the state represented in Figure 1 is an absorbing state. Since players do not conform to the same pattern of behavior, the absorbing state is not a convention. The absorbing state does, however, show the presence of local clusters of players who conform, expect others to conform, and want to conform (given that others conform) to the same pattern of behavior. We refer to the presence of such clusters in an absorbing state as coexistence of conventions. It is not hard to see that there are typically many absorbing states which exhibit such clusters. Note that in order for such a state to be an absorbing state the clusters of A-players and B-players should be at least of size $\frac{b}{\gamma}$ and $\frac{a}{\gamma}$, respectively, the off-diagonal payoffs must be less than γ and finally, the population must be large enough so that the population could not become the interaction neighborhood for a player, i.e., $\lfloor \frac{n}{2} \rfloor > \max \left\{ \frac{a}{\gamma}, \frac{b}{\gamma} \right\}$. ♦

The following theorem specifies the condition on the size of the population under which convergence of adaptive play without mistakes can be guaranteed almost surely. Let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to x.

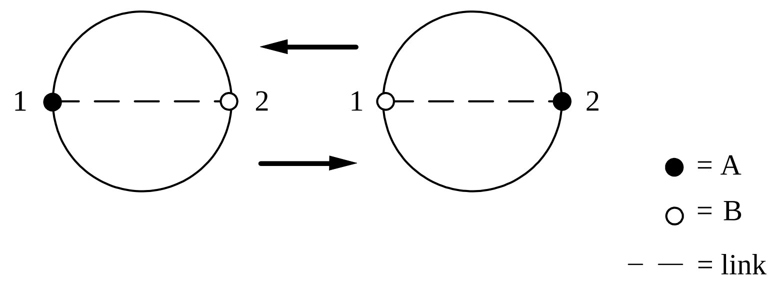

Figure 2: A cycle

Theorem 2 *Assume that* $\left\lfloor \frac{n}{2} \right\rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. *Then for any initial state* $s_0 \in \mathcal{S}$ *adaptive behavior without mistakes converges almost surely to an absorbing state.*

The proof of Theorem 2 is contained in Section 7.1.

The condition $\left\lfloor \frac{n}{2} \right\rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$ requires that the population is sufficiently large to ensure that for every player $i \in N$ there exist a player $j \neq i$ such that, independent of the outcome of the game, it is too costly for them to be linked. According to Theorem 2 this condition excludes the possibility that the dynamic process ends up in a cycle. The necessity of this assumption is illustrated with Example 3.

Example 3 Consider a population consisting of two players who are situated on a circle and assume that $\gamma = 1$ and $\min\{c, d\} \geqq 1$, i.e., player 1 and player 2 want to be linked independent of the outcome of the game. The two players will therefore form a link (if not already present) in the first stage of period 1 and sustain it in all subsequent periods. Since the players adapt their actions myopically as specified by (2), it can easily be verified that any initial state such that the actions of player 1 and player 2 are different, results in a cycle as represented in Figure 2, where both players will adapt their pure strategies continuously. Consequently, the players' pure strategies will be different every period and adaptive behavior without mistakes will never settle down. ♦

4 Adaptive Play with Mistakes

In this section we consider *adaptive play with mistakes*. With both kinds of mistakes as part of the model, i.e., $\tau > 0$ and $\varepsilon > 0$, the transition matrix of the Markov process is *irreducible* and *aperiodic*. Irreducibility means that for every pair of states s, $s' \in \mathcal{S}$, there exists a time $l := l(s, \tilde{s})$ such that $\left(P^l\right)_{s\tilde{s}} > 0$, i.e., every state in the state space can be reached from every other state with positive probability. A Markov process is aperiodic if for every state s in the state space it holds that $P_{ss} > 0$, i.e., for every state there is a positive probability of remaining there in the next period.

A *stationary distribution* of the Markov process is a row vector $\phi := \phi(\tau, \varepsilon) \in \Delta_{|\mathcal{S}|-1}$ such that $\phi P = \phi$, where

$$\Delta_{|\mathcal{S}|-1} = \left\{ \nu \in \mathbb{R}^{|\mathcal{S}|} \,\middle|\, \nu_s \geqq 0 \text{ for all } s \in \mathcal{S} \text{ and } \textstyle\sum_{s \in \mathcal{S}} \nu_s = 1 \right\}.$$

In Lemma 7 found in Section 7.2 we summarize the standard results in the literature that an irreducible and aperiodic Markov process is ergodic, meaning that the Markov process has a unique stationary distribution, and that the process converges to this stationary distribution from any initial state. Furthermore, along any sample path the distribution of realized states approaches the stationary distribution almost surely.

As mentioned earlier, the stationary distribution is a good description of the behavior of the adaptive process in the long run. Namely, the adaptive process jumps from an absorbing state to the basin of attraction of another absorbing state due to the occurrence of very unlikely realizations of the perturbations of the dynamic process, i.e., of the players' mistakes. Only given a very long period of time, these jumps will occur often enough to produce a well-defined stationary distribution, which represents an asymptotic probability distribution over the states.

To explore what the stationary distribution of adaptive play with mutations looks like, we introduce the following notation. An x-tree t on $\mathcal{S} = \mathcal{G} \times \mathcal{A}$ is a function $t : \mathcal{S} \to \mathcal{S}$ such that $t(x) = x$ and for all $s \neq x$ there exists an m with $t^m(s) = x$. The stationary distribution $\phi(\tau, \varepsilon)$ is characterized by

$$\phi_x(\tau, \varepsilon) = c(\tau, \varepsilon) \sum_{t \in H_x} \prod_{s \neq x} P_{st(s)}(\tau, \varepsilon), \tag{5}$$

where H_x is the set of x-trees on $\mathcal{S}$ and $c(\tau, \varepsilon)$ is a continuous function in τ and ε. Notice that Foster and Young [10], Kandori, Mailath and Rob [17], and Ellison [8] use the same characterization of the stationary distribution. We refer to those papers and to Freidlin and Wentzell [11] for background material on the characterization.

Now consider the transition matrix $P(\tau, \varepsilon)$ describing adaptive behavior with mistakes. The transition probabilities $P_{s\tilde{s}}(\tau, \varepsilon)$, i.e., the entries of the transition matrix, are continuous in τ and ε. In fact, the transition probabilities are either zero or given by a polynomial in τ and ε. The constant term of such a polynomial is nonzero if and only if the transition $s \to \tilde{s}$ occurs with positive probability in the Markov process describing adaptive behavior without mistakes ($\tau = \varepsilon = 0$). Furthermore, since each time period only two players can alter their (potential) mutual link and adapt their actions, the polynomial is of τ-order and ε-order at most 1 and 2, respectively.

Because we are interested in small probabilities of τ-mistakes and ε-mistakes, we consider the asymptotic behavior of the stationary distribution $\phi(\tau, \varepsilon)$ as $\tau \to 0$ and $\varepsilon \to 0$. Henceforth, we denote the behavior of the stationary distribution $\phi(\tau, \varepsilon)$ as $\tau \to 0$ and $\varepsilon \to 0$ by $\lim_{\tau,\varepsilon\to 0} \phi(\tau, \varepsilon)$. In the analysis as stated in this section, we assume that whenever these limits are taken, τ and ε go to zero at the same rate, i.e.,

$$0 < \lim_{\tau,\varepsilon\to 0} \frac{\tau}{\varepsilon} < \infty. \tag{6}$$

The importance of this assumption will be discussed in detail in Section 5.

The asymptotic stationary distribution gives the percentage of the time that the Markov process will spend in any state in the long run. The following theorem states that the $\left(A, \frac{a}{\gamma}\right)$-convention is the unique stochastically stable state, which means that when the two mistake probabilities go to zero at the same rate, adaptive play with mistakes will be in the $\left(A, \frac{a}{\gamma}\right)$-convention almost all the time, provided that the population is sufficiently large. Notice that the structure of the proof of Theorem 4 is identical to the structure of the proof of part (a) of Theorem 1 in Ellison [8].

Theorem 4 *Let $n \geqq 3$. If $0 < \lim_{\tau,\varepsilon\to 0} \frac{\tau}{\varepsilon} < \infty$, then $\lim_{\tau,\varepsilon\to 0} \phi_{\left(A, \frac{a}{\gamma}\right)}(\tau, \varepsilon) = 1$.*

The proof of Theorem 4 is contained in Section 7.2.

The intuition behind the theorem above is similar to the corresponding results in Kandori, Mailath and Rob [17], Young [24], and Jackson and Watts [15]. Namely, the $\left(A, \frac{a}{\gamma}\right)$-convention or, equivalently, the risk-dominant convention is the state with the largest basin of attraction, meaning that the minimal numbers of τ-mistakes and ϵ-mistakes needed to escape from its basin of attraction are larger than the minimal numbers of mistakes needed to leave the basin of attraction of any other state.

5 Discussion and Simulations

We now discuss the robustness of the model presented in Section 2. In particular, we focus on alternative specifications of the stochastic and deterministic part of the dynamic process and discuss how this would influence the results.

The stochastic part of the dynamic process not only consists of the mistake probabilities, but also of the random selection of a pair of players. First, consider the selection of a pair of players. Notice that this is the only stochastic element of adaptive play without mistakes, as analyzed in Section 3. The presence of this random element drives the convergence result as stated in Theorem 2. Without this random part of the model, i.e., with all players myopically updating links and pure strategies each period, cycles can not be excluded as shown by Ellison [8].

Random selection of a pair of players is not the only way to obtain a convergence result in a model without mistake probabilities. An alternative way to establish such a result is analyzed by Young [24] and applied by Jackson and Watts [15], where the random element is originated in a sampling procedure by the players. Young [24] assumes that players base their decisions on limited information about the actions of other players in the recent past. More precisely, every player inspects k plays drawn without replacement from the most recent $m \geqq k$ periods. A convergence result in the absence of mistake probabilities can be obtained in case the fraction $\frac{k}{m}$, which measures the completeness of the players' information, is small enough. In other words, convergence can only be ensured when the degree of randomness is sufficiently high. Results similar to those formulated in Theorem 2 and Theorem 4 could therefore still be obtained when we rephrase the present model such that it fits in the framework as analyzed by Young [24] and Jackson and Watts [15]. Alternatively, we could allow for the identified pair of players not only to alter their (potential) mutual link but also to possibly sever all other links they are involved in, without changing the results.

Second, consider the stochastic part of the dynamic process reflecting that players occasionally make mistakes when implementing their decisions. In Theorem 4, we assumed that τ and ε go to zero at the same rate. The specification of the model states that we assume the two kinds of mistakes to be independent of the state of the process. Bergin and Lipman [3] show that this assumption is of 'crucial' importance, meaning that allowing for τ-mistake probabilities or ε-mistake probabilities to differ across states and in particular to go to zero at different rates would change the results significantly. Indeed, by choosing the mistake probabilities appropriately, the $\left(B, \frac{b}{\gamma}\right)$-convention may even become the unique stochastically stable state. Van Damme and Weibull [6], however, restore the result that in 2×2 coordination games the risk-dominant equilibrium will be selected in the long run by considering endogenously determined mistake probabilities. More precisely, they consider a model where players can control the probability of making a mistake at some costs. As players will try harder to avoid mistakes leading to larger payoff losses, the mistake probabilities depend on the state of the process. They show that mistake probabilities nevertheless go to zero at the same rate when the costs become negligible.

Finally, we make two remarks concerning the deterministic part of the dynamic process. First, instead of assuming that a player can observe the pure strategies of all potential links independently of whether or not he is linked with them, we could alternatively consider the case that a player is only able to observe the pure strategies of the players he is currently linked with. The decision to establish a new link could then be based on comparing the maintenance costs with the expected payoff in the coordination game, where the expected payoff is given by a player's average gross benefit in the last period. We conjecture that coexistence of conventions in the long run will no longer be possible when this specification of the deterministic part of the dynamic process is used. To motivate this point, notice that the state illustrated in Figure 1 will no longer be a steady state of adaptive play without mistakes. Namely, adjacent A-players and B-players will establish a link if given the opportunity. As can be concluded from the proof of Theorem 4, the result that the $\left(A, \frac{a}{\gamma}\right)$-convention is the unique stochastically stable state of adaptive play with mistakes remains true for this alternative specification.

Second, the deterministic part of the process is characterized by sequential decision making of the players. In the first stage of a period players are concerned with the network formation process and in the second stage players decide upon their pure strategies in the coordination game. Considering players who decide on links and pure strategies simultaneously would, however, not significantly change any of the results. Convergence to an absorbing state of adaptive play without mistakes can still be obtained because simultaneous decision making does not decrease the degree of randomness in the model. Furthermore, adaptive play with mistakes will still converge to the risk-dominant convention as the size of the basins of attraction does not change significantly.

We adapt a measure introduced by Ellison [8] to illustrate the extent to which play resembles its long run limit. The long run limit with mistakes is characterized by the $\left(A, \frac{a}{\gamma}\right)$-convention. For the simulations reported below, we only examine play as it resembles its long run limit in the action of players and leave the graphical examination for future research. We denote the number of players playing A at time k as $A(a_{i,k})$ and note that this does not indicate the extent to how nearly the interaction network resembles its long run limit. $W(N, \tau, \varepsilon, \alpha)$ is the expected waiting time until at least $1 - \alpha$ of the players simultaneously play A given that everyone starts off playing B, i.e.,

$$W(N, \tau, \varepsilon, \alpha) = E(\min\{k | A(a_{i,k}) \geqq (1 - \alpha)N\} | a_{i,0} = B).$$

Tables I lists the observed values of $W(N, \tau, \varepsilon, \alpha)$ indicated by $W^R(N, \tau, \varepsilon, \alpha)$ for the endogenous interaction model such that all players initially play B and that the initial network is randomized, i.e., the starting condition is determined by a random network. In Table I-A we consider the coordination game described by $a = 2$, $b = 1$, and $c = d = 0$. In Table I-B we consider the coordination game given by $a = 4$, $b = 5$, $c = 3$, and $d = 0$. (We refer to the first set of payoffs as Game 1 and the second set of payoffs as Game 2.) Both sets of payoffs induce a player to play strategy A if at least $\frac{1}{3}$ of his interaction neighborhood play A. In both games A is the risk dominant equilibrium but in Game 2 B is Pareto superior to A. This is not the case in Game 1.

	$\tau = \varepsilon = 0.025$	$\tau = \varepsilon = 0.05$	$\tau = \varepsilon = 0.1$
$N = 10$	71.262 [4705] (71.31)	43.853 [33.5] (33.467)	30.591 [28] (15.705)
$N = 20$	527.999 [510] (304.428)	272.438 [233.5] (195.144)	105.405 [73] (90.489)
$N = 50$	3762.491 [3713] (699.923)	3340.933 [3253.5] (704.659)	1181.522 [1129] (563.383)
$N = 100$	19600.22 [19553] (1973.643)	17876.0 [17826] (2481.601)	11132.06 [11201.5] (2387.776)

Table I-A: Observed $W^R(N, \tau, \varepsilon, 0.25)$ for Game 1

In Table I-A we denote by N the population size, ε the probability of a mistake in the selection of actions, and τ the probability of a mistake in link formation. All waiting times reported are until 75% of the players play A simultaneously. The mean waiting times are reported first, the median waiting times are given in square brackets, while the standard deviation is given in parentheses.

	$\tau = \varepsilon = 0.025$	$\tau = \varepsilon = 0.05$	$\tau = \varepsilon = 0.1$
$N = 10$	* * *	221.108 [35] (442.426)	39.926 [30] (48.626)
$N = 20$	* * *	947.117 [676] (975.1461)	140.79 [78] (145.0575)
$N = 50$	* * *	5066.152 [4861.5] (1739.067)	1637.692 [1598] (819.478)
$N = 100$	* * *	* * *	16792.38 [16494.5] (4798.067)

Table I-B: Observed $W^R(N, \tau, \varepsilon, 0.25)$ for Game 2.

The waiting times reported in Table I-B are generated under similar conditions as those reported in Table I-A. In Table I-B the indication * refers to waiting times that average more than 30,000 iterations. These long waiting times are not feasible within the software application.

Tables II-A and II-B list the observed values of $W^E(N, \tau, \varepsilon, \alpha)$ for the endogenous interaction model such that initially all players play the risk dominated convention

B and the initial interaction network is the empty one.

	$\tau = \varepsilon = 0.025$	$\tau = \varepsilon = 0.05$	$\tau = \varepsilon = 0.1$
$N = 10$	8.69 [7] (11.302)	9.904 [8] (9.596)	13.058 [10] (11.05)
$N = 20$	14.127 [13] (3.608)	15.902 [15] (5.385)	23.464 [19] (18.104)
$N = 50$	35.769 [35] (5.611)	40.611 [39] (8.656)	58.183 [50] (27.342)
$N = 100$	68.087 [68] (7.222)	77.774 [76] (12.09)	106.75 [101] (29.77)

Table II-A: Observed $W^E(N, \tau, \varepsilon, 0.25)$ for game 1

Again the reported waiting times are until 75% of the players play A. The mean number of iterations is reported first, followed by the median number of iterations in square brackets, and the standard deviation in parentheses.

For the observed number of iterations with initially the empty network are reported in Table II-B:

	$\tau = \varepsilon = 0.025$	$\tau = \varepsilon = 0.05$	$\tau = \varepsilon = 0.1$
$N = 10$	39.783 [7] (445.963)	11.791 [8] (30.827)	14.071 [10] (11.554)
$N = 20$	14.95 [13] (5.465)	17.074 [15] (9.054)	22.259 [19] (11.345)
$N = 50$	35.733 [35] (5.63)	41.459 [39.5] (13.231)	58.44 [51] (28.348)
$N = 100$	68.31 [67] (7.244)	76.695 [75] (11.597)	108.957 [101] (33.203)

Table II-B: Observed $W^E(N, \tau, \varepsilon, 0.25)$ for game 2.

The difference between the number of observed iterations with a random network — reported in Tables I-A and I-B — and an empty network — reported in Tables II-A and II-B — is caused by the inertia in the network formation process. Because the deterministic part of the process is characterized by a sequential decision making process, on average the random network will have too many links that are relatively costly and, therefore, have to be severed. However, until those links are severed, they are part of the interaction neighborhoods of the players involved. This is causes a higher number of iterations necessary to converge the evolutionary process.

In general, as the mistake probabilities increase, the dispersion of the expected waiting time decreases, as does the expected waiting time. Both of these observations are consistent with what we expect. The more likely a mistake occurs, the more likely the process will enter an absorbing state for the $\left(A, \frac{a}{\gamma}\right)$-convention. Considering that the deterministic part of the process is characterized by sequential decision making of the players, it is not surprising that convergence seems dependent on network size. Consider that at a minimum $\frac{n(n-1)}{2}$ periods must pass for each pair of players to meet. As a rough proxy we divide the observed average number of iterations by the number of possible links. For example, from the reported values in Table II-A for Game 1 initiated with an empty network and mistake probabilities $\tau = \varepsilon = 0.025$ we compute

$$\begin{aligned}
\frac{W^E(10,\tau,\varepsilon,0.25)}{45} &= 1.58, \\
\frac{W^E(20,\tau,\varepsilon,0.25)}{190} &= 2.78, \\
\frac{W^E(50,\tau,\varepsilon,0.25)}{1225} &= 3.07, \\
\frac{W^E(100,\tau,\varepsilon,0.25)}{4950} &= 3.96.
\end{aligned}$$

These values indicate the average number of times all links are queried for at least $\frac{3}{4}$ of all players to play the risk dominant convention A given that everyone initially plays B. It is not surprising that the average number of times each pair needs to be queried increases with the number of players. This is caused by the assumption that all relevant pairs are queried in a random order. Another reason for the increase in the waiting times is that perhaps a pocket of the risk dominated equilibrium persists and coexists for a substantial amount of time. The probability of such an event would increase with the number of players.

Finally, we tested whether the threshold of a convergence rate of 75% is a reasonable proxy for complete convergence. We performed a set of simulations where $\alpha = 0$, meaning that the measured number of iterations were for the stopping rule that 100% of the players play A when initially all players play B. The observed number of iterations on average only exceeded the reported values in Tables I-A, I-B, II-A and II-B by approximately half the number of potential links $\frac{n(n-1)}{2}$.

6 Concluding Remarks

We have analyzed a large population coordination game where not only the actions of the players but also the communication network is subject to evolutionary pressure. Cost considerations of social interaction are incorporated by exploiting the spatial structure of the model, i.e., the costs of interacting increase in the distance between two players on the circle.

The Markov process without mistakes describing medium-run behavior is shown to converge to an absorbing state, which may be characterized by coexistence of conventions. In the long-run, when mistakes are possible with small but nonvanishing probabilities, coexistence of conventions is no longer possible as the risk-dominant convention is the unique stochastically stable state. These results require that $c, d \geqq \gamma$. If $c, d < \gamma$, i.e., mismatched pairs obtain net losses when making links with their closest neighbors, we conjecture that we would obtain a similar insight as the second main theorem (Proposition 2) of Jackson and Watts [15]. That is, we would expect to see both coordination equilibria as stochastically stable states.

Interesting topics for future research are as follows. First, considering a larger class of games, see e.g. Kandori, Mailath and Rob [17] and Young [24], in the framework of the present paper. Second, extending the model to also allow for players who are indirectly connected to play a game as is common practice in the literature on network formation. When value is given to indirect connections, other types of networks could be generated with perhaps other implications.

7 Proofs

7.1 Proof of Theorem 2

To explore the conditions under which adaptive play without mistakes converges to an absorbing state, we develop the two lemmas. We define the set $\tilde{S} \subset \mathcal{S}$ by

$$\tilde{S} := \left\{ s \in \mathcal{S} \;\middle|\; a_i \in \arg\max_{\tilde{a}_i} \pi_i \left(\tilde{a}_i, a_{\mathcal{L}_i}\right) \text{ for all } i \in N \right\},$$

i.e., the set $\tilde{S}$ contains all states $s \in \mathcal{S}$ such that all players $i \in N$ choose a pure strategy that results in the highest possible gross benefit, given the current pure-strategy profile of the players in their neighborhood. Lemma 5 specifies a condition on the size of the population under which adaptive play without mistakes becomes contained in $\tilde{S}$ with positive probability in a finite period of time. Let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to x.

Lemma 5 *Assume that* $\left\lfloor \frac{n}{2} \right\rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. *Then for any state* $s \in \mathcal{S}$ *there is a probability* $p_s > 0$ *that adaptive play without mistakes reaches a state* $\tilde{s} \in \tilde{S}$ *in* $K_s < \infty$ *periods.*

Proof. Consider a state $s \in \mathcal{S}$ and a player $i \in N$. With positive probability player i is matched with a player $j \in N$ such that $i \neq j$ and $c_{ij} > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Notice that for every player $i \in N$, existence of such a player j is guaranteed because we assume that $\left\lfloor \frac{n}{2} \right\rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Obviously, players i and j will never establish a mutual link. Furthermore, if a mutual link was already present, then it will definitely be severed. After that, both players will adapt their pure strategy and, as they no longer belong to each other's neighborhood, end up choosing a pure strategy that results in the highest possible benefit given the current pure strategies of the players in their neighborhood.

Consider the positive probability event of successively matching all players $i \in N$ in the way specified above. Notice that player i changing his pure strategy may imply that players in his neighborhood, who have already undergone the above procedure, no longer choose a pure strategy that results in the highest possible benefit given the current pure-strategy profile in their neighborhood. We restore this characteristic for the involved players by means of the positive probability event of once again matching all of them in the same pairs as before. Obviously, this may in turn result in other players, who have already undergone the above procedure, no longer playing optimally given the current pure-strategy profile in their neighborhood, and so on. It therefore remains to be shown that such a restoration process will terminate in finite time.

Suppose player i changes his pure strategy and becomes an A-player. As the restoration process does not include any link formation or link severance and $a - c > d - b$, the change by player i makes pure strategy A more attractive for all players involved in the restoration process. Consequently, all involved A-players will definitely remain A-players, while involved B-players possibly become A-players. Since we have a finite number of players such a restoration process terminates in finite time. A similar argument holds when player i becomes a B-player. ■

Lemma 6 in turn states that, starting from a state which is contained in $\tilde{S}$, adaptive play without mistakes converges to an absorbing state with positive probability in a finite period of time.

Lemma 6 *For any state $s \in \tilde{S}$ there is a probability $q_s > 0$ that adaptive play without mistakes converges to an absorbing state in $L_s < \infty$ periods.*

Proof. Consider a state $s \in \tilde{S}$ and a player $i \in N$. Suppose player i is an A-player, i.e., $a_i = A$. With positive probability player i is successively matched in pairs with all players $j \in N$ such that $j \in \mathcal{L}_{i,k}$, $a_j = B$, and $c_{ij} > \frac{c}{\gamma}$. Since the costs c_{ij} are strictly larger than the payoff $\frac{c}{\gamma}$ to player i, all links between player i and these players j will be severed. Because $b > c$ this dynamic process of link severance implies that $\pi_i \left(A, a_{\mathcal{L}_{i,k},k-1}\right)$ decreases less than $\pi_i \left(B, a_{\mathcal{L}_{i,k},k-1}\right)$, causing player i to remain an A-player. Furthermore, all players j remain B-players (when their link with player i is severed) due to the fact that $a > d$ causes $\pi_j \left(B, a_{\mathcal{L}_{j,k},k-1}\right)$ to decrease less than $\pi_j \left(A, a_{\mathcal{L}_{j,k},k-1}\right)$.

We continue with the positive probability event that player i is successively matched in pairs with all players j such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} > \frac{a}{\gamma}$. Since costs c_{ij} exceed payoffs $\frac{a}{\gamma}$ for these pairs of players, all links between player i and players j will be severed. Notice that at some point during this dynamic process of link severance, player i may become a B-player because $a > d$, i.e., $\pi_i\left(A, a_{\mathcal{L}_{i,k},k-1}\right)$ decreases more than $\pi_i\left(B, a_{\mathcal{L}_{i,k},k-1}\right)$. First, consider the case that player i is an A-player when he severs the link with a player j. Since $a > d$, the severance of this link may cause player j to become a B-player. This in turn may cause that A-players in the neighborhood of player j also want to become B-players, and so on. It can easily be verified that successively matching all the involved players in pairs, which is a positive probability event, results in a dynamic process only consisting of players changing their pure strategy from A to B and possibly severing their mutual link. Obviously, such a dynamic process will terminate in a finite number of steps and result in a state contained in $\tilde{S}$. Notice that player i may also be one of the players who changes his pure strategy from A to B in the above dynamic process.

Second, consider the case that player i, because of one of the two reasons mentioned above, becomes a B-player while he is severing links with the players j. With positive probability player i is successively matched in pairs with all players $j \in N$ such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} > \frac{d}{\gamma}$. Since $a > d$, this includes all players j such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} > \frac{a}{\gamma}$. All these links will be severed because the costs c_{ij} are strictly larger than the payoff $\frac{d}{\gamma}$ to player i. Furthermore, player i and players j who are involved in this dynamic process will not change their pure strategy because $a > d$ and $b > c$, respectively.

The above argument shows that a player i, who is initially an A-player, will become an A-player such that

$$\begin{cases} \left[a_j = A \text{ and } c_{ij} > \frac{a}{\gamma}\right] \Rightarrow j \notin \mathcal{L}_i, \\ \\ \left[a_j = B \text{ and } c_{ij} > \frac{c}{\gamma}\right] \Rightarrow j \notin \mathcal{L}_i, \end{cases} \tag{7}$$

for all $j \in N$ with $i \neq j$, or a B-player such that

$$\begin{cases} \left[a_j = A \text{ and } c_{ij} > \frac{d}{\gamma}\right] \Rightarrow j \notin \mathcal{L}_i, \\ \\ \left[a_j = B \text{ and } c_{ij} > \frac{b}{\gamma}\right] \Rightarrow j \notin \mathcal{L}_i, \end{cases} \tag{8}$$

for all $j \in N$ with $i \neq j$, with positive probability and in finite time. A similar argument can be used to show that this also holds for a player i who is initially a B-player. In fact, starting from any state $s \in \tilde{S}$, we attain a state $\tilde{s} \in \tilde{S}$ such that all players $i \in N$ simultaneously satisfy either (7) or (8) with positive probability and in finite time. Such a state can actually be attained by successively applying the above dynamic process to all players $i \in N$.

Notice that the interactive effects may cause that players at some point in time no longer satisfy (7) or (8) even though the dynamic process has already been applied to them. For these players the dynamic process has to be repeated. However, due to the fact that the dynamic process does not include any link formation and, in case no links are severed, only consists of players changing their pure strategy in the same way, this can only happen a finite number of times.

Now consider a state $\tilde{s} \in \tilde{S}$ such that all players satisfy either (7) or (8) and there exists a player $i \in N$ such that $a_i = A$. With positive probability player i is successively matched in pairs with all players $j \in N$ such that $j \notin \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} \leqq \frac{a}{\gamma}$. Obviously, not only will all these pairs of players establish a link, but also will none of the involved players change his pure strategy. We continue with the positive probability event that player i is successively matched in pairs with all players $j \in N$ such that $j \notin \mathcal{L}_{i,k}$, $a_j = B$, and $c_{ij} \leqq \frac{c}{\gamma}$. Again, all pairs of players will establish a link, which may possibly cause players to change their pure strategy. In fact, players i and players j may become B-players and A-players, respectively. As explained before, however, every time that link severance, or link formation for that matter, causes a player to change his pure strategy, the subsequent dynamic process of changing pure strategies by other players will terminate in finite time with positive probability. In case player i actually becomes a B-player, we continue with the positive probability event that player i is successively matched in pairs with all players j such that $j \notin \mathcal{L}_{i,k}$, $a_j = B$, and $c_{ij} \leqq \frac{b}{\gamma}$, and all players j such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $d < \gamma \cdot c_{ij} \leqq a$, which obviously only results in link formation and link severance, respectively.

The above argument shows that a player i, who is an A-player in a state $\tilde{s} \in \tilde{S}$, will become an A-player such that

$$\begin{cases} a_j = A \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{a}{\gamma} \right], \\ \\ a_j = B \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{c}{\gamma} \right], \end{cases} \tag{9}$$

for all $j \in N$ with $i \neq j$, or a B-player such that

$$\begin{cases} a_j = A \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{d}{\gamma} \right], \\ \\ a_j = B \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{b}{\gamma} \right], \end{cases} \tag{10}$$

for all $j \in N$ with $i \neq j$, with positive probability and in finite time. A similar argument can be used to show that this also holds for a player i who is a B-player in a state $\tilde{s} \in \tilde{S}$. Furthermore, starting from a state $\tilde{s} \in \tilde{S}$, we attain a state such that all players $i \in N$ simultaneously satisfy (9) or (10) with positive probability and in finite time. In that case, no single players wants to adapt his pure strategy or sever any of the links he is involved in, and no pair of players wants to establish a mutual link, which implies that we have attained an absorbing state of the Markov process.

Like before it may happen that in order to attain an absorbing state from a state $\tilde{s} \in \tilde{S}$, the dynamic process described above has to be applied repeatedly to some players. However, because of the following two reasons this can only happen a finite number of times. First, the part of the dynamic process dealing with link severance does not result in players changing their pure strategies. Second, parts of the dynamic process that do consist of players changing their pure strategies will always terminate in finite time with positive probability as they only involve changes in the same direction. ■

Now, we are able to prove Theorem 2 which specifies the condition on the size of the population under which convergence of adaptive play without mistakes can be guaranteed almost surely.

Theorem 2 *Assume that* $\left\lfloor \frac{n}{2} \right\rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$*. Then for any initial state* $s_0 \in S$ *adaptive behavior without mistakes converges almost surely to an absorbing state.*

Proof. Lemma 5 states that for any state $s \in \mathcal{S}$ there is a probability $p_s > 0$ that adaptive play without mistakes will be given by a state $\tilde{s} \in \tilde{S}$ in $K_s < \infty$ periods, provided that $\left\lfloor \frac{n}{2} \right\rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. According to Lemma 6, for any state $s \in \tilde{S}$ there is a probability $q_s > 0$ that adaptive play without mistakes converges to an absorbing state in $L_s < \infty$ periods.

Note that $\mathcal{S}$ is finite. Let $p = \min_{s\in\mathcal{S}} p_s > 0$, $q = \min_{s\in\tilde{S}} q_s > 0$, $K = \max_{s\in\mathcal{S}} K_s < \infty$, and $L = \max_{s\in\tilde{S}} L_s < \infty$, it follows that there exists a positive integer $M = K + L < \infty$, and a positive probability $r = pq > 0$, such that from any initial state $s_0 \in \mathcal{S}$, the probability is at least r that adaptive play without mistakes converges to an absorbing state within M periods, provided that $\left\lfloor \frac{n}{2} \right\rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Obviously, both M and r are time-independent and state-independent. Let m be an arbitrary integer. Hence, the probability of not reaching an absorbing state after at least mM periods is at most $(1-r)^m$. This implies that this probability goes to zero as $m \to \infty$. ■

7.2 Proof of Theorem 4

Lemma 7 *Assume that the transition matrix* P *is irreducible and aperiodic. Then the stationary distribution* ϕ *is unique. Furthermore, for any* $\nu \in \Delta_{|\mathcal{S}|-1}$*,*

$$\nu P^k \to \phi \quad \text{as } k \to \infty.$$

Also, for all initial states $s_0 \in S$*,*

$$\frac{1}{K}\sum_{k=1}^{K} X_s(s_k) \to \phi_s \quad \text{almost surely as } K \to \infty,$$

where

$$X_s(s_k) = \begin{cases} 1 & \text{if } s_k = s, \\ 0 & \text{otherwise.} \end{cases}$$

(See Theorem 1.2 *and Theorem* 1.3 *in Chapter* 3 *of Karlin and Taylor [18])*

We write $r(z)$ for a successor of state z in the Markov process describing adaptive play without mistakes. Obviously, $r(z)$ is not uniquely determined since it depends on which pair of players can alter their (potential) mutual link and adapt their pure strategies. For this reason we write

$$D\left(A, \frac{a}{\gamma}\right) = \left\{ z \;\middle|\; \lim_{m\to\infty} \Pr\left(r^m(z) = \left(A, \frac{a}{\gamma}\right)\right) = 1 \right\}$$

for the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$-convention. Hence, the basin of attraction $D\left(A, \frac{a}{\gamma}\right)$ consists of the states that will eventually be taken to the $\left(A, \frac{a}{\gamma}\right)$-convention by adaptive play without mistakes.

To determine what the stationary distribution looks like when mistake probabilities go to zero, we need the following two lemmas. Lemma 8 gives upper bounds for the number of τ-mistakes and ε-mistakes that are needed to enter the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$-convention with positive probability.

Lemma 8 *Let $x \notin D\left(A, \frac{a}{\gamma}\right)$, $y_1, ..., y_H \notin D\left(A, \frac{a}{\gamma}\right)$, and $y \in D\left(A, \frac{a}{\gamma}\right)$. There exists a transition from x to y via $y_1, ..., y_H$ such that*

$$P_{xy_1}(\tau, \varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h y_{h+1}}(\tau, \varepsilon) \cdot P_{y_H y}(\tau, \varepsilon) > 0, \tag{11}$$

where (11) is of τ-order and ε-order at most 0 *and* n*, respectively.*

Proof. To prove Lemma 8 it suffices to show that $D\left(A, \frac{a}{\gamma}\right)$ can be reached from any state with positive probability using at most n ε-mistakes. First, let the population size n be even. Consider the positive probability event of successively matching players i and $i+1$ in pairs, where $i = 2j-1$ and $j = 1, ..., \frac{n}{2}$. Every time a player has the opportunity to update his pure strategy, he chooses A. Obviously, this requires at most n ε-mistakes and leaves us with a state $y \in D\left(A, \frac{a}{\gamma}\right)$.

Second, let the population size n be odd. Consider the positive probability event of successively matching players i and $i+1$ in pairs, where $i = 2j-1$ and $j = 1, ..., \frac{n-1}{2}$. Again, all players involved in this positive probability event choose pure strategy A whenever they have the opportunity to update their strategy. This requires at most $n-1$ ε-mistakes and leaves us with a state such that $a_i = A$ for all $i = 1, ..., n-1$. Now, we continue with the positive probability event of matching player n with an A-player $i^* \in \{1, ..., n-1\}$, where player i^* is an A-player who is linked to at least one other A-player. Notice that existence of such an A-player can always be established without the need for any additional mistakes. Then, at most one more ε-mistake is needed for player n to become an A-player, while player i will remain an A-player. Again, we have reached a state $y \in D\left(A, \frac{a}{\gamma}\right)$ using at most n ε-mistakes. ■

Lemma 9 gives a lower bound for the total number of mistakes in a certain product of transition probabilities. All transition probabilities contained in the product refer to transitions which are specified by an x-tree t, with $x \notin D\left(A, \frac{a}{\gamma}\right)$, and which start from a state in the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$-convention.

Lemma 9 *Let $n \geqq 3$. The sum of the τ-order and ε-order of*

$$\prod_{s \in D\left(A, \frac{a}{\gamma}\right)} P_{st(s)}(\tau, \varepsilon) > 0,$$

with t an x-tree such that $x \notin D\left(A, \frac{a}{\gamma}\right)$, is at least $n+1$.

Proof. First, consider the state $s = \left(A, \frac{a}{\gamma}\right) \in D\left(A, \frac{a}{\gamma}\right)$. Obviously, any path starting in the $\left(A, \frac{a}{\gamma}\right)$-convention and eventually leaving $D\left(A, \frac{a}{\gamma}\right)$ contains at least one ε-mistake.

Second, consider a state s such that (i) every player $i \in N$ is linked with his two adjacent neighbors, i.e., every player $i \in N$ is linked with all players $j \neq i$ such that

$$\min\{|j-i|, n-|j-i|\} = \gamma,$$

and (ii) there is exactly one player i such that $a_i = B$. Obviously, such a state s is contained in $D\left(A, \frac{a}{\gamma}\right)$ when $n \geqq 3$ and there exist exactly n different states that satisfy the conditions (i) and (ii). Furthermore, any path starting in such a state s and eventually leaving $D\left(A, \frac{a}{\gamma}\right)$ (or reaching another state satisfying conditions (i) and (ii)) contains at least 1 mistake (either a τ-mistake or an ε-mistake). Hence, we know that the sum of the number of τ-mistakes and ε-mistakes in $\prod_{s \in D\left(A, \frac{a}{\gamma}\right)} P_{st(s)}(\tau, \varepsilon)$ is at least $n+1$. ■

Theorem 4 *Let $n \geqq 3$. If $0 < \lim_{\tau,\varepsilon \to 0} \frac{\tau}{\varepsilon} < \infty$, then $\lim_{\tau,\varepsilon \to 0} \phi_{\left(A, \frac{a}{\gamma}\right)}(\tau, \varepsilon) = 1$.*

Proof. The characterization as specified by (5) allows us to express the quantity $\frac{\phi_x(\tau,\varepsilon)}{\phi_{\left(A, \frac{a}{\gamma}\right)}(\tau,\varepsilon)}$ as a ratio of polynomials in τ and ε for any state x. To prove Theorem 4 it is sufficient to show that for n sufficiently large, i.e., $n \geqq 3$, it holds that

$$\lim_{\tau,\varepsilon \to 0} \frac{\phi_x(\tau, \varepsilon)}{\phi_{\left(A, \frac{a}{\gamma}\right)}(\tau, \varepsilon)} = 0$$

for all $x \neq \left(A, \frac{a}{\gamma}\right)$. This will follow if we demonstrate that for any x-tree t $\left(x \neq \left(A, \frac{a}{\gamma}\right)\right)$ such that $\prod_{s \neq x} P_{st(s)}(\tau, \varepsilon) > 0$, we have

$$\lim_{\tau, \varepsilon \to 0} \frac{\prod_{s \neq x} P_{st(s)}(\tau, \varepsilon)}{\sum_{t' \in H_{\left(A, \frac{a}{\gamma}\right)}} \prod_{s \neq \left(A, \frac{a}{\gamma}\right)} P_{st'(s)}(\tau, \varepsilon)} = 0.$$

This in turn follows if we show that there exists an (A, a)-tree t' such that

$$\prod_{s \neq \left(A, \frac{a}{\gamma}\right)} P_{st'(s)}(\tau, \varepsilon) > 0$$

and

$$\lim_{\tau, \varepsilon \to 0} \frac{\prod_{s \neq x} P_{st(s)}(\tau, \varepsilon)}{\prod_{s \neq \left(A, \frac{a}{\gamma}\right)} P_{st'(s)}(\tau, \varepsilon)} = 0. \tag{12}$$

We show that (12) holds by distinguishing two cases.

First, assume that $x \in D\left(A, \frac{a}{\gamma}\right)$. Define t' by

$$t'(z) = \begin{cases} r(z) & \text{if } z \in D\left(A, \frac{a}{\gamma}\right), \\ t(z) & \text{otherwise.} \end{cases}$$

Notice that t' is an $\left(A, \frac{a}{\gamma}\right)$-tree because for any state z the path described by t' initially coincides with t and hence eventually enters $D\left(A, \frac{a}{\gamma}\right)$. From the first point at which $t'^m(z) \in D\left(A, \frac{a}{\gamma}\right)$, the tree maps every point to a successor according to adaptive play without mistakes and hence reaches $\left(A, \frac{a}{\gamma}\right)$. In this case the ratio specified in (12) equals

$$\frac{P_{\left(A, \frac{a}{\gamma}\right) t\left(\left(A, \frac{a}{\gamma}\right)\right)}(\tau, \varepsilon) \prod_{s \in D\left(A, \frac{a}{\gamma}\right) - \left\{\left(A, \frac{a}{\gamma}\right), x\right\}} P_{st(s)}(\tau, \varepsilon)}{P_{xr(x)}(\tau, \varepsilon) \prod_{s \in D\left(A, \frac{a}{\gamma}\right) - \left\{\left(A, \frac{a}{\gamma}\right), x\right\}} P_{sr(s)}(\tau, \varepsilon)}.$$

The above expression converges to 0 as $\tau, \varepsilon \to 0$ because $P_{st(s)}(\tau, \varepsilon) / P_{sr(s)}(\tau, \varepsilon)$ is bounded, $P_{\left(A, \frac{a}{\gamma}\right) t\left(\left(A, \frac{a}{\gamma}\right)\right)}(\tau, \varepsilon) \to 0$, and $P_{xr(x)}(\tau, \varepsilon) \to \xi > 0$.

Second, assume that $x \notin D\left(A, \frac{a}{\gamma}\right)$. Define t' by

$$t'(z) = \begin{cases} r(z) & \text{if } z \in D\left(A, \frac{a}{\gamma}\right), \\ t(z) & \text{if } z \notin D\left(A, \frac{a}{\gamma}\right), \, z \neq x, \text{ and } z \neq y_1, ..., y_H, \\ y_1 & \text{if } z = x, \\ y_2 & \text{if } z = y_1, \\ ... & ... \\ y_H & \text{if } z = y_{H-1}, \\ y & \text{if } z = y_H, \end{cases}$$

where $y_1, ..., y_H \notin D\left(A, \frac{a}{\gamma}\right)$ and $y \in D\left(A, \frac{a}{\gamma}\right)$ are as specified in Lemma 5.5. Obviously, t' is again an $\left(A, \frac{a}{\gamma}\right)$-tree. In this case the ratio specified in (12) equals

$$\frac{\prod_{s \in D\left(A, \frac{a}{\gamma}\right)} P_{st(s)}(\tau, \varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h t(y_h)}(\tau, \varepsilon)}{P_{xy_1}(\tau, \varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h y_{h+1}}(\tau, \varepsilon) \cdot P_{y_H y}(\tau, \varepsilon) \cdot \prod_{s \in D\left(A, \frac{a}{\gamma}\right) - \left\{\left(A, \frac{a}{\gamma}\right)\right\}} P_{sr(s)}(\tau, \varepsilon)}.$$

Because of Lemma 8 and the fact that $P_{sr(s)}(\tau, \varepsilon) \to \mu > 0$, the denominator is of τ-order 0 and of ε-order at most n. According to Lemma 9, the numerator is of a higher total order if $n \geqq 3$. Consequently, the above expression converges to 0 as long as $0 < \lim_{\tau, \varepsilon \to 0} \frac{\tau}{\varepsilon} < \infty$. ■

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