Local Conventions in Game Play in an Evolving Dual Social Network Framework*

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Abstract

People usually perform economic interactions within the social setting of a small group, while they obtain relevant information from a broader source. We capture this feature with a dynamic interaction model based on two separate social networks. Individuals play a coordination game in an *interaction network*, while updating their strategies using information from a separate *influence network* through which information is disseminated. In each time period, the interaction and influence networks co-evolve, and the individuals' strategies are updated through a modified naïve learning process. We show that both network structures and players' strategies always reach a steady state, in which players form fully connected groups and converge to local conventions. We also analyze the influence exerted by a minority group of strongly opinionated players on these outcomes.

JEL: D70, D83, D85, C63, A14, L14

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1 Naïve learning and game play

In daily life economic agents interact with each other in a variety of ways. In many situations, economic value generating interactions are separated from communicative interactions and information sharing activities. Thus, economic agents interact with different agents for economic purposes than they do to learn about behavioral choices. In this paper we consider a theory that explicitly separates information sharing from game play into two spheres: an *interaction* network that describes how players play a coordination game with each other and a social information sharing or *influence* network where players learn about the strategies played by and the success of other players. We consider a dynamic learning process in which the interaction network, the influence network, and the selected strategies are all updated sequentially. As a consequence, in our theory, the interaction and influence networks are fundamentally distinct, but shown to be highly correlated.

Previous work has considered interaction networks separately from influence networks. Studies on the formation of interaction networks assumes that these networks guide game play only. Many authors adopt an evolutionary game-theoretic framework to study such game play on networks. In the most sophisticated of the developed models, players choose a strategy in a coordination game as well as their interaction partners, resulting into the endogenous emergence of the interaction network. The main insights from the models presented in, e.g., Droste et al. (2000), Mailath et al. (2001), Jackson and Watts (2002), Ely (2002), Goyal and Vega-Redondo (2005) and Ehrhardt et al. (2008) establish that global conventions might arise for a variety of evolutionary network formation mechanisms. Based on the interaction costs certain interaction structures might emerge, although rarely there emerge cliques with local conventions.

A second category of research considers influence networks and the process of information sharing only. A substantial number of contributions have been made toward the so-called French-DeGroot naïve learning mechanism as proposed by French (1956), Harary (1959) and DeGroot (1974). This model is in nature a simple Markov process based on the principle that individuals base their position on weighted averages of observed characteristics of other players. The French-DeGroot theory mainly addresses opinion formation in given social networks. The standard insight is that a global opinion emerges unless the influence network is partitioned or disjointed. Recently this theory has been extended further by Friedkin and Johnsen (1990, 1997), DeMarzo et al. (2003) and Golub and Jackson (2010). These extensions do not alter the conclusions from this type of model in a significant way. This is different if one introduces time-variation of the influence weights as developed in Hegselmann and Krause (2002), Weisbuch et al. (2002) and Pan (2010).[1]

The updating of our dual network structure requires the use of a single updating mecha-

[1] For a more fully developed overview we refer to the excellent surveys by Jackson (2007, 2009).

nism that addresses how the interaction network, the influence weights as well as the players' strategic choices are updated sequentially. We accomplish this by extending the French-DeGroot naïve learning mechanism rather than the evolutionary game-theoretic framework. This implies that all spheres in our dual network framework are updated on a naïve perception of the observed performance of one's neighbors in the influence network: Interactions are engaged in if they result in payoffs that exceed the given interaction costs; players are influenced more by successful neighbors; and they emulate strategies of these more successful neighbors.

Therefore, by the nature of the French-DeGroot model, in our approach it is assumed that players learn about strategies played by others and can freely observe their performance. Subsequently, each player attaches weights to her observations based on the payoffs as a measurement of success that the observed neighbors make. Based on these weights, a player then plays a strategy that is the weighted average of her neighbors' strategies.[2] We emphasise that in this process a player always takes account of her own choice and performance in the previous round of interaction. So, this implies that a successful player would put a higher weight on her own past performance rather than that of her less successful neighbors. Nevertheless, we prove later that the steady state is very rarely a Nash equilibrium.

We emphasize that strategies are updated using the influence weights in the standard French-DeGroot updating rule. This requires the strategy space in the coordination game to be *continuous*. Within this setting we show that the application of the French-DeGroot updating rule implies that the resulting steady state networks are *highly correlated*; and, in the emerging steady states, players interact with the *same* players that inform their strtategic position through the influence network.

Our choice of updating mechanism is justified by two arguments. First, the naïve learning mechanism can be viewed as a naïve form of a best response dynamic. Players focus on matching their choices on successful neighbors in the influence network. The better the performance of a neighbor in the interaction network, in particular with common interaction partners, the higher the influence weight assigned to that particular neighbor. This induces the player to emulate successful interaction with common interaction partners. So, strategies are essentially updated based on performance-weighted observations as would a player do in an evolutionary mechanism founded on principles of better reply rationality and imitation. For similar arguments we refer to Golub and Jackson (2010).

Finally, we note that the French-DeGroot naïve learning mechanism has some strong

[2]Here, we denote a player as a "neighbor" in the influence network of another player if that other player attaches a positive influence weight to observations made about that particular player. This implies that if one player is a neighbor of another, it might not be the case that the other player is a neighbor of that player; neighboring in the influence network is assumed to be a non-symmetric property, unlike that in the interaction network.

properties: It has a simple nature; it has strong convergence properties; and it characterizes influence in an intuitive fashion that corresponds to the underlying network structure (Jackson, 2007). These properties are particularly important in light of the complexity of our model, including the modification of multiple social interaction spheres as well as strategic updating.

Second, at this time there is no obvious implementation of more rational evolutionary processes of information sharing and learning that form a powerful alternative for the basic French-DeGroot rule. This omission implies that to extend evolutionary mechanisms to the complexity of multiple interaction spheres as well as strategic updating would likely result in rather convoluted a theory.

Within our model we establish some powerful insights that conform with insights from some evolutionary approaches as well as the most prevalent naïve learning models of opinion formation. First, the steady state strategic convention under conformism is in most cases *not* a Nash equilibrium. This fact is also reported by Blume and Easley (2006): Naïve learning is limited in nature and does not necessarily converge to a steady state satisfying rational expectations.

Furthermore, we are able to prove convergence results for arbitrary interaction cost levels. These descriptions fully characterize the resulting steady states consisting of a steady state interaction network, a steady state influence network, and a limit distribution of strategies played in the implemented coordination game.

For sufficiently high costs, there emerges a fully non-interactive state in which all players remain completely autarkic. In this state there is no interaction and no information sharing. This is naturally expected within the framework studied here.

For low enough interaction costs, on the other hand, there emerges a fully interactive state in which players conform to a global convention in their game play. All players interact with all other players and attach equal influence weights to all players. The emerging global convention is any strategy in the convex, continuous strategy subspace spanned by the strategies initially present in the population. As such the steady state equilibrium convention is not necessarily a Nash equilibrium and has no normative properties. This is akin to the opinion formation insights established in, e.g., DeGroot (1974), Golub and Jackson (2010), and Pan (2010).

For intermediate levels of interaction costs, there emerge cliques which adhere to local conventions. All members of a clique interact with each other and attach equal influence weights to each other, but do not interact with anybody outside the clique. Furthermore, the emerging local conventions are unique in the sense that cliques adhere to different strategic conventions. This implies that all cliques form communities that establish uniquely local conventions akin to Tiebout's theory of clubs (Tiebout, 1956). The established local conformism is also in line with some of the main insights from Kandori et al. (1993), Oechssler

(1999), Ely (2002), and Ehrhardt et al. (2008).

Following Pan (2010), we also consider the effects of the introduction of so-called *persistent* players. A player is considered to be "persistent" if throughout the dynamic updating and naïve learning process she does not modify the strategy that was initially assigned to her. This corresponds to assuming that these players have immutable opinions or strategies in their interaction with others. We assume here that other players still interact with a persistent player and are influenced by her, but that the persistent player is not observing and learning from other players in a meaningful way. Therefore, persistent players influence others, but are not influenced at all by any of their neighbors.

In the presence of persistent players, we arrive at the similar convergence results as discussed for the case without persistent players. However, we see that persistent players have extraordinary influence on the emerging strategic conventions: The (local) conventions emerging are identified as convex combinations of the persistent members of the clique *only* rather than convex combinations of all players in the clique. This implies the very strong insight that a single persistent player can completely determine the emerging local convention in his clique. This extraordinary power of persistent agents is also addressed by Acemoglu et al. (2010), Fagyal et al. (2010), and Pan (2010).

Acemoglu et al. (2010) studied a diffusion model with "forceful" individuals that are similar to our persistent players. Aside from the fact that their model focuses on the spread of information that is not in a game play context, there are two key differences between their work and the model in this paper. First, in their paper, the network structure is fixed and exogenous. Here our players choose to have a connection with a persistent player or not. Second, they employ a Bayesian learning model, which is very different from our naïve learning process.

The remainder of this paper is structured as follows. The next section introduces the formal setup of the model and the updating process. Section 3 analyzes the outcomes from the standard setting, while Section 4 discusses the influence of persistent players in our framework. Finally, Section 5 draws some conclusions for future directions of research. All proofs are relegated to an appendix.

2 A network model of social interaction

We consider a finite set of players $N = \{1, 2, \ldots, n\}$ who engage in playing a specified continuous strategy coordination game in an endogenous interaction network. We explicitly restrict these players' rationality in that they do not optimize over their strategy set; instead, the players select a strategy by weighing the information collected through an information network on the success of other players and the strategies that they use.

Next, we introduce a description of the social interaction network, the influence network, and subsequently formulate the dynamic interaction process.

2.1 The social interaction network

Each player $i \in N$ selectively builds social relationships with other players. The resulting *interaction network* at time $t \in \mathbb{N}$ is represented by an $n \times n$ adjacency matrix $\mathbf{G}^t$ with

$$\mathbf{G}^t_{ij} = \begin{cases} 1 & \text{if } i, j \text{ are connected,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

This implies that $\mathbf{G}^t_{ij} = \mathbf{G}^t_{ji}$ for all $i, j \in N$ and all t, i.e., the interaction network $\mathbf{G}^t$ is symmetric. Define $L^t_i = \{j \in N \mid \mathbf{G}^t_{ij} = 1\}$ as the set of player i's *neighbors* in the interaction network $\mathbf{G}^t$ at time t. Throughout, we assume that every player is always connected with herself, i.e.; $\mathbf{G}^t_{ii} = 1$, for all $i \in N$ and all t.

A subset $M \subseteq N$ of players is *connected* at time t if there exists a path between any 2 players $i, j \in M$. That is, there exist distinct players $i_1, \ldots, i_k \in M$ such that $i_1 = i$, $i_k = j$, and $\mathbf{G}^t_{i_m i_{m+1}} = 1$ for all $m \in \{1, \ldots, k-1\}$. A subset $M \subseteq N$ of players are in a *component* at time t if M is connected at time t and $\mathbf{G}^t_{ij} = 0$ for all $i \in M$ and $j \notin M$.

Throughout we assume that every connection in $\mathbf{G}^t$ is consent-based, which means that permission from both players is required for a link to be formed. On the other hand, a single player can always sever any link under her control.[3]

The decision on forming and severing links is based on the assumption that interaction is costly, i.e., both the initiation and the maintenance of a link between two players imposes the same cost $c \geq 0$ on both interacting parties. This implies that, when a link is initiated, both players pay the common interaction cost c. Also, each player pays the common interaction cost c for the maintenance of every existing link during each time period t. We emphasize that here we assume that each player $i \in N$ has no costs of interacting with herself.

A player $i \in N$ only interacts with her neighbors $j \in L^t_i$ at time $t \in \mathbb{N}$. The association between each pair of connected players is modelled as a continuous strategy, symmetric coordination game of social choices. In this game, each players chooses her standing in a social issue. Naturally, we assume that players with closer standings find their interaction more pleasant, thus the adoption of a coordination game.

Formally, the symmetric coordination game is presented by (P, π), where $P = [0, 1]$ is the common continuous strategy set[4] and $\pi \colon P^2 \to \mathbb{R}_+$ is a common non-negative payoff function for all $p_i, p_j \in P$. The strategy vector $\mathbf{p}^t = (p^t_1, \ldots, p^t_n)^{\mathrm{T}} \in [0, 1]^n$ presents all

[3] This is akin to the concept of pairwise stability seminally introduced in Jackson and Wolinsky (1996) and the standard stability concept in matching markets (Roth and Sotomayor, 1990).

[4] We interpret these strategies to be *positions* of both players in a value-generating interaction. Extreme positions would by represented by $p = 0$ and $p = 1$, while any $0 < p < 1$ signifies an "intermediate" position.

players' strategies at time t. The payoff function is defined by

$$\pi\left(p_i, p_j\right) = ap_ip_j + (1 - p_i)(1 - p_j), \tag{2}$$

where $a \geq 1$. For convenience we usually write $\pi_{ij}^t = \pi\left(p_i^t, p_j^t\right)$.

Due to the symmetric nature of the coordination game, each pair of players $i, j \in N$ receives identical payoffs from such coordination, i.e., $\pi_{ij}^t = \pi_{ji}^t$, for all $i, j \in N$.

The given game has three symmetric Nash equilibria:

- $p^A = \frac{1}{a+1}$ with payoff $\pi^A = \pi\left(\frac{1}{a+1}, \frac{1}{a+1}\right) = \frac{a}{a+1} \leq 1$;
- $p^B = 0$ with payoff $\pi^B = \pi(0, 0) = 1$;
- and $p^C = 1$ with payoff $\pi^C = \pi(1, 1) = a \geq 1$.

It is clear that $\pi^A \leq \pi^B \leq \pi^C$. Also note that with $a \geq 1$, taking the neutral stand of $p_i^t = 0.5$ is never a best response.

2.2 Information dissemination

In our approach information dissemination is separated from actual game play that takes place in the interaction network $\mathbf{G}^t$. Within our framework, information sharing is modelled as the observation of other players' actions and attaching a weight to these observations. This weight indicates how much influence one player has over another. Therefore, we formally model the information sharing process as a set of influence weights. We later show that the updating of weight assignment represents players' attempt to maximize their payoffs in a myopic and bounded rational way.

Formally, for every time period $t \in \mathbb{N}$ the information sharing system is introduced as an $n \times n$ nonnegative matrix $\mathbf{T}^t$ which we refer to as the *influence matrix* at time t. For all $i, j \in N$, the number $\mathbf{T}_{ij}^t \in [0, 1]$ indicates the weight that player i places on player j's strategic choice at time t. A higher weight indicates that a player weighs the other's choice of her strategy higher. Thus, the influence matrix $\mathbf{T}^t$ captures the information collection process at time t.

We assume that for every $t \in \mathbb{N}$ the influence matrix $\mathbf{T}^t$ is row-stochastic, i.e., the influence weights sum up to unity for each player $i \in N$:

$$\sum_{j=1}^{n} \mathbf{T}_{ij}^t = 1 \text{ and } \mathbf{T}_{ij}^t \geq 0, \text{ for all } i, j \in N, \text{ for all } t \in \mathbb{N}. \tag{3}$$

Unlike the interaction network $\mathbf{G}^t, \mathbf{T}^t$ may be asymmetric, so that $\mathbf{T}_{ij}^t \neq \mathbf{T}_{ji}^t$ for some i, j. Moreover, one's information collection is not restricted to one's neighbors in the interaction network.[5] Also, for some i, j, it can hold that $\mathbf{T}^t_{ij} > 0$ while $\mathbf{G}^t_{ij} = 0$. On the other hand, $\mathbf{G}^t$ and $\mathbf{T}^t$ are correlated via the learning process, as will be discussed in the next subsection.

2.3 The updating process

As stated in the introduction of this paper, the updating process is a model of a naïve learning dynamic based on the French-DeGroot updating rule. This process consists of the subsequent updating of the interaction network, the influence network, and the players' strategies based on the payoffs collected and observed.

We initialize the updating process by assigning random strategies and influence weights. The initial interaction network is assumed to be fully autarkic, i.e., every player only interacts with herself.

After initialization, two players are randomly selected and each selected player updates her interaction network based on mutual consent; subsequently, all players update their influence weights based on observations in the information-sharing network; and, finally, all players update their strategies and play the game with their partners in the interaction network.

Schematically, this updating process is represented as the flow diagram depicted in Figure 1. Next, we specify the details of each step in the updating process.

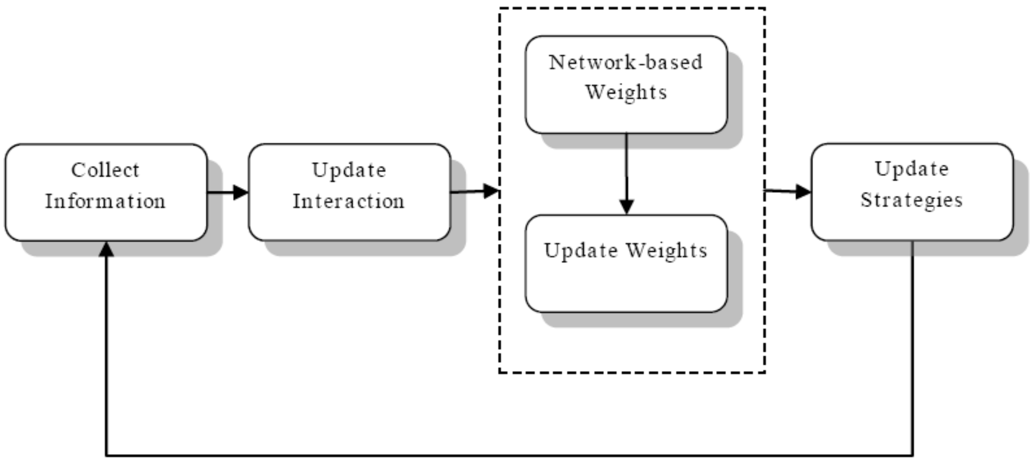


Figure 1: Updating process in our dual network framework

[5]By the nature of French-DeGroot learning, essentially we assume complete observation, i.e., one can observe strategies of all players. However, one may not assign a positive weight on everyone that she observes. Therefore, we define that one's neighbors in the influence network are those that she assigns positive weights to.

The initialization

At $t = 0$ we have an initial coordination structure $\mathbf{G}^0$ and an initial influence matrix $\mathbf{T}^0$. The initial coordination structure is assumed to be autarkic, i.e.;

$$\mathbf{G}^0_{ii} = 1, \text{ for all } i \in N \text{ and } \mathbf{G}^0_{ij} = 0 \text{ for all } i \neq j. \tag{4}$$

As for the initialization of the information network and the strategies, we assume that players initially are assigned arbitrary strategies and an arbitrary influence distribution. That is,

$$p^0_i \in [0, 1], \text{ for all } i \in N. \tag{5}$$

$$\sum_{j=1}^{n} \mathbf{T}^0_{ij} = 1 \text{ for all } i \in N; \quad \mathbf{T}^0_{ij} \in [0, 1], \text{ for all } i, j \in N. \tag{6}$$

Note that, although $\mathbf{G}^0_{ii} = 1$ for all i, the case that $\mathbf{T}^0_{ii} = 0$ is not excluded.

Updating the interaction network

During the updating process in period t, two players i and j are randomly selected with uniform probability to consider their connectivity. The link ij will be formed (if the two are not connected) or maintained (if they are connected already) if and only if for both of them, the payoffs from the link *strictly* exceed the interaction cost. That is, the updating rule for the interaction network $\mathbf{G}^t$ is given by

$$\mathbf{G}^t_{ij} = \begin{cases} 1 & \text{if } \pi^{t-1}_{ij} > c \\ 0 & \text{if } \pi^{t-1}_{ij} \leq c \end{cases} \tag{7}$$

and

$$\mathbf{G}^t_{hk} = \mathbf{G}^{t-1}_{hk} \text{ for all } (h, k) \neq (i, j) \tag{8}$$

Note that the payoff between strategies 0 and 1 is zero. Therefore, a player i with $p^t_i = 0$ and a player j with $p^t_j = 1$ never choose to form a link when given the chance. However, depending on the cost, the two may be connected via a mutual player whose strategy is between 0 and 1.

Updating the influence network

After the two randomly selected players i and j update their interaction relationship $\mathbf{G}^t_{ij}$ described above, all players update their weight assignments in the influence network $\mathbf{T}^{t-1}$. We model the updating of the influence matrix $\mathbf{T}^{t-1}$ to be based on the observed payoffs from the game play in the updated $\mathbf{G}^t$. The principle is that a player's partners or neighbors in the interaction network $\mathbf{G}^t$ act as effective filters for more beneficial links and higher payoffs.

Links are formed and maintained based on a cost-benefit evaluation. Therefore for a player, the connectedness between one of her neighbors and another player implies a reasonable potential for collecting high payoffs from coordinating with her neighbor's partner. When an player decides on how much influence weight to place on another player, she calculates the total payoffs that her neighbors could obtain from associating with that player, given all players' current actions and connectivity. This observation procedure is carried on among all players, while the selected connectivity shows its impact on processing the collected information.

We recall that the influence matrix is row-stochastic. Thus, each player redistributes her influence weight assignment proportionally according to the total payoffs and then normalizes the weights to make sure that the row sum equals to unity. This implies that the redistribution of influence can be modelled as

$$\mathbf{T}^t_{ij} = \frac{w^t_{ij}}{\sum_{k=1}^{n} w^t_{ik}}, \text{ for all } i, j \in N \text{ and } t \in \mathbb{N}, \tag{9}$$

where

$$w^t_{ij} = \sum_{l \in L^t_i} \mathbf{G}^t_{lj} \pi^{t-1}_{lj}.$$

Here $\mathbf{T}^t_{ij}$ is the weight assigned by i to j. If j's action does not guarantee a sufficiently high payoff, j would not be connected with any of i's neighbors. That is, $\mathbf{G}^t_{kj} = 0$ for all $k \in L^t_i$. Consequently, $w^t_{ij} = 0$, which results to zero weight $\mathbf{T}^t_{ij} = 0$. That is, a player does not place any weight on someone who does not provide the potential to generate high enough payoffs.

Also, we note that if $L^t_i = \emptyset$, we have a problem when applying the equations above in that all weights w^t_{ij} are 0 and the sum of the i-th row of the influence matrix $\mathbf{T}^t$ does not add up to 1. This problem is prevented by the assumption that each player is connected with herself during initialization at $t = 0$ and stays connected with herself during any subsequent period $t \in \mathbb{N}$, since it is assumed that player i has no costs related to her self-referential coordination $\mathbf{G}^t_{ii} = 1$. That is, $L^t_i \neq \emptyset$ for all i, t because we have at least $i \in L^t_i$.

For those who get zero weights in the influence matrix, their actions and information from them do not count when player i updates her strategy p^t_i. In other words, each player actually takes the weighted average among the beneficial or potentially beneficial actions during the updating process.

Strategy updating and game play

Finally, all players update their strategies based on the information collected in period $t-1$. Using the updated influence matrix $\mathbf{T}^t$, all players determine their strategies using the French-DeGroot updating rule. That is,

$$p_i^t = \sum_{j \in N} \mathbf{T}_{ij}^t p_j^{t-1} \qquad \text{for all } i \in N, t > 0. \tag{10}$$

So the updating process for all players can be conveniently written as:

$$\mathbf{p}^t = \mathbf{T}^t \mathbf{p}^{t-1}. \tag{11}$$

After updating her strategy, each player $i \in N$ plays the given coordination game with her neighbors $j \in L_i^t$ in $\mathbf{G}^t$ and collects payoffs for the period t. Subtracting her interaction costs for all active links (except the one with herself) in period t we get payoffs

$$\pi_i^t = \sum_{h \in L_i^t} \pi_{ih}^t - (|L_i^t| - 1)\, c = \sum_{h \in L_i^t} [a p_i^t p_h^t + (1 - p_i^t)(1 - p_h^t)] - (|L_i^t| - 1)\, c, \tag{12}$$

where $|L_i^t|$ is size of the set L_i^t, i.e., the number of i's neighbors during time period t.

The time period t ends when game play is completed. The process repeats in the next period $t+1$.

3 Convergence of behavior

In the process described above, players update their interaction neighborhood, influence weights, and strategies in a myopic manner in that they do not consider the implications of updating to the future. Rather, they base their decisions on the success of the strategies adopted by the players that they observe. Our examination of the convergence of this learning process follows studies such as Ellison and Fudenberg (1993, 1995); Bala and Goyal (1998); Banerjee and Fudenberg (2004); Lorenz (2005); and Golub and Jackson (2010).

Definition 3.1 *Consider the updating process* $(\mathbf{G}, \mathbf{T}, \mathbf{p})$ *introduced in the previous section.*

(i) $\mathbf{G}^t$ *is **convergent** if there exists* t^**, such that* $\mathbf{G}_{ij}^t = \mathbf{G}_{ij}^{t^*}$ *for all* $t > t^*$.

(ii) $\mathbf{p}^t$ *is **convergent** if there exists* p_i^* *for all* i *such that for all* $\epsilon > 0$*, there exists* t^* *such that* $|p_i^t - p_i^*| < \epsilon$ *for all* $t > t^*$.

(iii) $\mathbf{T}^t$ *is **convergent** if there exists* $\mathbf{T}_{ij}^*$ *for all* i, j *such that for all* $\epsilon > 0$*, there exists* t^* *such that* $|\mathbf{T}_{ij}^t - \mathbf{T}_{ij}^*| < \epsilon$ *for all* $t > t^*$.

We further introduce some auxiliary concepts with regard to describing the resulting steady state. In particular, a set of players $M \subseteq N$ is a *limit component* in the given dynamics $(\mathbf{G}, \mathbf{T}, \mathbf{p})$, if there is some $\hat{t} \in \mathbb{N}$ such that M is a component in $\mathbf{G}^t$ for all $t > \hat{t}$.

Note here we do not require the sub-network defined by $\mathbf{G}^t$ on M to be convergent. In other words, there can be different paths between a *specific* pair of players in M in $t_1, t_2 > \hat{t}$, as long as there exists a path between *any* pair of players in the set at all $t > \hat{t}$.

The cost of coordination $c \geq 0$ is a critical factor that largely determines the interaction network structure, which in turn affects the updating of the influence weights and consequently the adjustment of strategies. The next proposition determines the relevant bounds for arbitrary levels of the interaction cost c and time moments $t \in \mathbb{N}$.

Proposition 3.2 *Let $c \geq 0$ be an arbitrary level of coordination cost and $(\mathbf{G}, \mathbf{T}, \mathbf{p})$ be the dynamic process as described by equations (7) to (11) based on an arbitrary initialization $\mathbf{p}^0$ and $\mathbf{T}^0$. For the strategy vector at every $t \in \mathbb{N}$ denote by $\underline{p}^t = \min\{p_1^t, \dots, p_n^t\}$ its lower bound and by $\overline{p}^t = \max\{p_1^t, \dots, p_n^t\}$ its upper bound. Furthermore, we define*

$$\underline{\pi}^t = \min\left\{a\left(\overline{p}^t\right)^2 + \left(1 - \overline{p}^t\right)^2,\ a\left(\underline{p}^t\right)^2 + \left(1 - \underline{p}^t\right)^2,\ a\underline{p}^t\overline{p}^t + (1 - \underline{p}^t)(1 - \overline{p}^t)^2\right\} > 0 \quad (13)$$

$$\overline{\pi}^t = \max\left\{a\left(\overline{p}^t\right)^2 + \left(1 - \overline{p}^t\right)^2,\ a\left(\underline{p}^t\right)^2 + \left(1 - \underline{p}^t\right)^2\right\} > 0. \quad (14)$$

Then the following properties hold:

(a) *$\underline{\pi}^t$ and $\overline{\pi}^t$ are a lower bound and an upper bound, respectively, for the set of all payoffs $\{\pi_{ij}^t \mid i, j \in N\}$.*

(b) *$\underline{\pi}^t$, is increasing, i.e., $\underline{\pi}^t \leq \underline{\pi}^{t+1}$.*

(c) *$\overline{\pi}^t$, is decreasing, i.e., $\overline{\pi}^t \geq \overline{\pi}^{t+1}$.*

It follows that $\overline{\pi}^0$ and $\underline{\pi}^0$ are an upper and lower bound for all π_{ij}^t for all $i, j \in N$, for all $t \in \mathbb{N}$. When we have a limit component (including the case where the whole population of players form a limit component), the upper and lower bounds of payoffs are *strictly* increasing and decreasing as long as strategies in the set are not uniform.

Corollary 3.3 *Let M be a limit component, we define $\underline{p}_M^t = \min_{i \in M}\{p_i^t\}$ and $\overline{p}_M^t = \max_{i \in M}\{p_i^t\}$. Furthermore, we define*

$$\underline{\pi}_M^t = \min\left\{a\left(\overline{p}_M^t\right)^2 + \left(1 - \overline{p}_M^t\right)^2,\right.$$
$$\left. a\left(\underline{p}_M^t\right)^2 + \left(1 - \underline{p}_M^t\right)^2,\ a\underline{p}_M^t\overline{p}_M^t + (1 - \underline{p}_M^t)(1 - \overline{p}_M^t)^2\right\} > 0 \quad (15)$$

$$\overline{\pi}_M^t = \max\left\{a\left(\overline{p}_M^t\right)^2 + \left(1 - \overline{p}_M^t\right)^2,\ a\left(\underline{p}_M^t\right)^2 + \left(1 - \underline{p}_M^t\right)^2\right\} > 0. \quad (16)$$

Then if $\overline{p}^t_M > \underline{p}^t_M$,[6] *there exists* $\overline{k} \geq 1$, *and* $\underline{k} \geq 1$, *such that* $\overline{\pi}^t_M > \overline{\pi}^{t+\overline{k}}_M$, *and* $\underline{\pi}^t_M < \underline{\pi}^{t+\underline{k}}_M$.

Next, Proposition 3.4 states that in the long run, players never learn to play a Nash equilibrium unless we have extreme initial conditions where $\mathbf{p}^0 = (0, \ldots, 0)^{\mathrm{T}}$ or $\mathbf{p}^0 = (1, \ldots, 1)^{\mathrm{T}}$. In particular, even if some of the players select a pure strategy initially, they will abandon that selection in favor of a purely strategy through the influence of other players. Proofs for Proposition 3.2 and 3.4 as well as Corollary 3.3 can be found in the appendix.

Proposition 3.4 *Let* $(\mathbf{G}, \mathbf{T}, \mathbf{p})$ *be the dynamic process as described by equations (7) to (11) based on an arbitrary initialization* $\mathbf{p}^0$ *and* $\mathbf{T}^0$. *If* $c < \underline{\pi}^0$, $n \geq 2$, *and there are* $i, j \in N$ *such that* $p^0_i \neq p^0_j$, *then there exists* $T \in \mathbb{N}$ *such that for all* $t > T:\ 0 < p^t_h < 1$ *for all* $h \in N$.

We are able to show that the updating process converges given any initial conditions and parameter settings, including cost level between lower and upper bounds of payoffs. At the steady state, players are either isolated or form "cliques" of fully connected groups of conforming players and the consensus is actually unique. Note that in some cases, the whole network forms a single clique.

Theorem 1 *For* $n \geq 2$, *let* $(\mathbf{G}, \mathbf{T}, \mathbf{p})$ *be the dynamic process as described by equations (7) to (11) based on an arbitrary initialization* $\mathbf{p}^0$ *and* $\mathbf{T}^0$. *Then there exists a partitioning* $\{N_1, \ldots, N_K\}$ *of* N *such that for every* $k \in \{1, \ldots, K\}$ *the set* N_k *satisfies the following properties:*

(a) N_k *is a limit component in* $(\mathbf{G}, \mathbf{T}, \mathbf{p})$ *with* $\lim_{t\to\infty} \mathbf{G}^t_{ij} = 1$ *for all* $i, j \in N_k$;

(b) $\mathbf{p}^t$ *converges on* N_k *to a unique local convention, i.e.; there is some* $p^*_k \in P$ *such that* $\lim_{t\to\infty} p^t_i = p^*_k$ *for all players* $i \in N_k$ *and for every player* $j \notin N_k$: $\lim_{t\to\infty} p^t_j \neq p^*_k$;

(c) *and* $\mathbf{T}^t$ *converges on* N_k *to a uniform distribution, i.e.; for every player* $i \in N_k$:

$$\lim_{t\to\infty} \mathbf{T}^t_{ij} = \begin{cases} \frac{1}{|N_k|} & \text{for every } j \in N_k \\ 0 & \text{for every } j \notin N_k \end{cases} \tag{17}$$

A proof of Theorem 1 can be found in the Appendix.

From the proof of Theorem 1 we derive two extreme cases. First, for low enough interaction costs, there results a single, giant clique in the population. Hence, all players converge to a global convention under uniform learning.

Corollary 3.5 *If* $c < \underline{\pi}^0$, *then the updating process converges to a situation in which there emerge a fully connected interaction network, evenly distributed influence weights, and all players play the same strategy* p^*.

[6] This is true if and only if there exist $i, j \in M$ such that $p^t_i \neq p^t_j$.

For sufficiently high interaction costs, we arrive at the antithesis of this situation in which all players remain autarkic.

Corollary 3.6 *If $c \geq \overline{\pi}^0$, then the updating process converges to a situation in which there emerges an autarkic interaction network, the influence matrix is an identity matrix, and all players adhere to their initial strategy p_i^0.*

The proofs for the Corollary 3.5 and 3.6 are straightforward and omitted here.

Sensitivity analysis

Above we have described the outcomes of the updating process. With the arbitrary initial weights and strategies, as well as the randomness in selecting a pair of players to update the interaction network during each period, when cost is in the medium range determined as $\underline{\pi} < c < \overline{\pi}$, any one of the 2^N different possible subsets of the player set N is likely to be one of the final cliques. Even when c is very low or very high and we are certain about the final network structures, the values of the local conventions are nothing deterministic due to the various randomness mentioned above. In this subsection we illustrate with the help of computer simulations.

In these simulations we set the highest possible payoff which can be obtain from playing (A, A) at $a = 2$, implying that the interaction cost c effectively ranges from 0 to 2. We take $\delta = 0.05$ as the increment in the cost level. At each cost level, the simulation is run 100 times with different initial configurations.

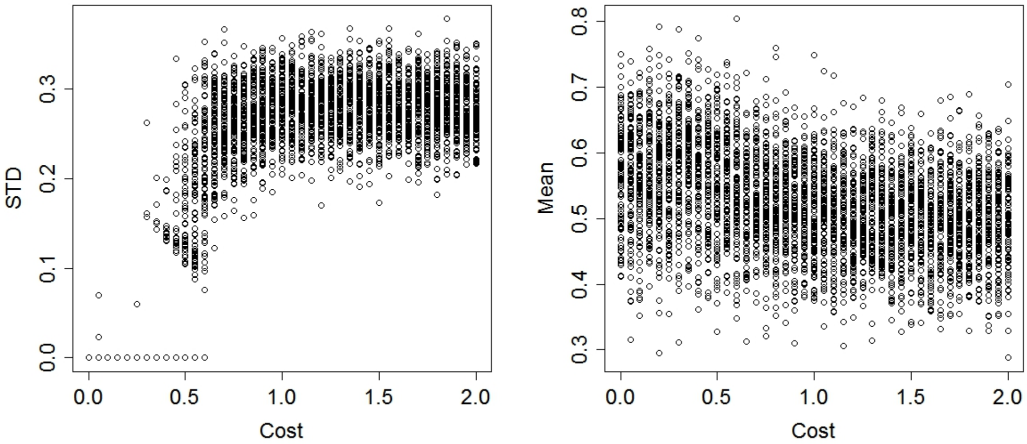
Different sizes of $n \in \{20, 40, 60, 80, 100\}$ are implemented. The change in society size does not show any significant effect on the final outcome. Therefore, in this subsection we only show the results for $n = 20$. Results are recorded when the learning process researches a steady state.[7]

In Figure 2 the x-axis shows value of cost c. In panel (a), the y-axis indicates the standard deviation among all strategies at the steady state. In panel (b), the y-axis shows the mean value of all strategies at the steady state.

When standard deviation equals 0, it means that the strategies fully conform. In panel (a) we see that standard deviation is 0 when cost is relatively low. However, even with a very low cost of 0.05, we see that some initial conditions do not lead to complete consensus.

The patterns in panel (a) with $0.2 \leq c \leq 0.6$ illustrate the randomness of outcomes the best, with the standard deviation scattering in a wide range. For higher costs, we expect smaller components to autarkic networks at the steady states, which implies that the final strategies are close or equal to the initial strategies. Recall that the initial strategies follow the uniform distribution between 0 and 1, the standard deviation and mean value of the final

[7]The program checks convergence every n periods by calculating the norm of the difference of the strategy vector $\Delta \mathbf{p}^t = \mathbf{p}^{t+n} - \mathbf{p}^t$. Convergence is detected when the norm of $\Delta \mathbf{p}^t$ equals 0.



(a) Standard deviation (b) Mean value

Figure 2: Cost sensitivity analysis of the basic model

strategies coincide with the patterns of the distribution indeed, i.e., the standard deviation is near 0.28 and the final strategies scatter uniformly.[8]

4 Introducing persistent players

Founded on the notion of persistency introduced in the learning model of Pan (2010), we extend the basic model to include persistent players. A player is called "persistent" if she does not change her initially assigned strategy over time. It was already shown in Pan (2010) that persistent players have a significant influence on the outcome of a French-DeGroot naïve learning process. We confirm this insight in our more elaborate dual network framework as well.

The set of persistent players is denoted by $S \subset N$. Here we consider persistent players with a common strategy as well as different strategies. We refer to the first case as a *uniform* group of persistent players and the second case a *diverse* group of persistent players. We assume that every persistent player updates her interaction network L^t_s, through which non-persistent players get influenced by their persistent peers. On the other hand, we assume that a persistent player also updates her influence weights, which practically play no role in the updating process. Therefore, we can construct an influence matrix where persistent players place full weight of 1 on themselves and zero-weight on everybody else. Namely, following the network and influence weight updating given by (7) to (9), we can rewrite the strategy updating for both uniform and diverse persistent players as:

$$\mathbf{p}^t = \tilde{\mathbf{T}}^t \mathbf{p}^{t-1}, \tag{18}$$

[8]Note here that the standard deviation for a uniform distribution between 0 and 1 is approximately 0.28.

where $\tilde{\mathbf{T}}^t_{ij} = \mathbf{T}^t_{ij}$ for all $i \notin S$, and $\tilde{\mathbf{T}}^t_{sj} = 1$ if $j = s$ and 0 otherwise, for all $s \in S$.

Convergence with persistent players

Our main insight is that the introduction of persistent players into the population alters the outcome of the social learning process significantly. Before we illustrate such control power, we first describe the general pattern of the steady state that applies to both types of persistent players. The following result is the analogue of Theorem 1 for the case of persistent players.

Theorem 2 *Let* $(\mathbf{G}, \mathbf{T}, \mathbf{p})$ *be a dynamic process described in (7) to (9) and (18). Then for any* $\mathbf{p}^0_{-S}$, $\mathbf{p}^0_S$, $\mathbf{T}^0$, $c \geq 0$, *and* $a \geq 1$, *there exists a partitioning* $\{N_1, \ldots, N_K\}$ *of* N *such that for every* $k \in \{1, \ldots, K\}$ *the set* N_k *satisfies the following properties:*

(a) N_k *is a limit component in with* $\mathbf{G}^t_{ij} = 1$ *for all* $i, j \in N_k \setminus S$;

(b) $\mathbf{p}^t$ *of non-persistent players in* N_k *converges to a local convention, i.e.; there is some* $p^*_k \in P$ *such that* $\lim_{t\to\infty} p^t_i = p^*_k$ *for all players* $i \in N_k \setminus S$;

(c) *and* $\mathbf{T}^t$ *converges to a unique distribution for every non-persistent member of* N_k, *i.e.; there exists some* $\widehat{\mathbf{T}}^k \in [0,1]^{N_k}$ *such that for every non-persistent member* $i \in N_K \setminus S$ *it holds that* $\lim_{t\to\infty} \mathbf{T}^t_{ij} = \widehat{\mathbf{T}}^k_j$ *for all* $j \in N_k$ *and* $\lim_{t\to\infty} \mathbf{T}^t_{ij} = 0$ *for all* $j \notin N_k$.

Theorem 2 appears to be very similar to Theorem 1 that describes the outcome without persistent players. We no longer have fully connected cliques. Instead, all non-persistent players in each component form a fully connected sub-clique. To understand this, recall that persistent players do not change their strategies. So there may exist persistent players i, j with fixed p_{si} and p_{sj} such that the payoffs from coordination are no larger than cost c. On the other hand, there may exists a fully connected clique of non-persistent players with strategy p_k such that both p_{si} and p_{sj} generate $\pi > c$ with p_k. In that case we have links between the two persistent players and the non-persistent players, but the component that includes them all are not a fully connected clique because the two persistent players never form a link between them.

The pattern of influence weights at the steady state is different. Namely, the influence weights within each clique N_k are not evenly distributed as it is in the previous case described in Theorem 1. Instead, they now exhibit identical rows. That is, each player in the group has the same weight distribution, but does not necessarily place the same weight on everyone.

For low enough cost, consider a non-persistent player being connected to more than one persistent players that might have different strategies. Then her payoffs from those diverse persistent players are different. Therefore the influence weights on those persistent players are also different. With the same reasoning, we see why non-persistent players also have the same column weights, which refer to the weights that they place on each of the non-persistent player in the group who has the same strategy in the end.

Next we consider the special case in which all persistent players have a common strategy that they adhere to. In this case, the clique with persistent players is pulled towards that particular strategy as shown in the next theorem.

Theorem 3 *Let* $(\mathbf{G}, \mathbf{T}, \mathbf{p})$ *be a dynamic process described in (7) to (9) and (18) such that there exists a uniform set of persistent players* $S \subset N$ *with a common persistent strategy given by* $p_s^0 = p_s^t = p_\alpha \in [0, 1]$ *for all* $t \in \mathbb{N}$*. Then for any* $\mathbf{p}_{-S}^0$, $\mathbf{p}_S^0$, $\mathbf{T}^0$, $c \geq 0$*, and* $a \geq 1$*, the following are true:*

(a) $\lim_{t\to\infty} p_i^t = p_\alpha$ *for all* i *such that* $\mathbf{G}_{is}^* = 1$ *for some* $s \in S$*, where* $\mathbf{G}^*$ *is the limit of* $\mathbf{G}^t$.

(b) *The uniform persistent players are either all isolated or all in the same limit component.*

(c) $\lim_{t\to\infty} \mathbf{T}_{ij}^t = \dfrac{1}{| L_i^* |}$ *for all* $i \notin S$*, for all* $j \in L_i^*$*, where* L_i^* *is* i*'s set of neighbors in* $\mathbf{G}^*$.

Theorem 3 states that the consensus within the clique that has all uniform persistent players equals the persistent players' initial strategy p_α. Especially, when $c < \underline{\pi}^0$ and $p_\alpha = 1$ or $p_\alpha = 0$, the social learning process converges to the Nash equilibrium outcomes (A, A) and (B, B), respectively. Also, in this case where the persistent players have uniform initial (persistent) strategies, the total number of them $|S|$ only affects the speed of convergence, not the final outcome.

Note that a clique without any uniform persistent players is also conforming to a common strategy, which is similar to the previous basic case and not determined by the persistent players' initial strategy. Besides, all cliques, with or without persistent players, feature uniform influence weight distribution.

Corollary 4.1 formulates the analogue of Corollary 3.5 for the case with uniform persistent players.

Corollary 4.1 *When* $c < \underline{\pi}$ *and there exists a uniform set of persistent players, the social learning process converges to a fully connected interaction network, evenly distributed weights, and all non-persistent players' strategies converge to* p_α.

Finally, we discuss the specifics of the convergence process if the persistent are not uniform, but rather diverse. In this case the persistent players select different strategies that they stubbornly adhere to. We arrive at the conclusion that in this case, cliques form that select a convex combination of the strategies adhered to by the persistent members of the clique:

Theorem 4 *Let* $(\mathbf{G}, \mathbf{T}, \mathbf{p})$ *be a dynamic process described in (7) to (9) and (18). Suppose that there exists a diverse set of persistent players* $S \subset N$ *with* $2 \leq |S| \leq n - 1$ *such that* *there are $s_i, s_j \in S$ with $p^0_{s_i} \neq p^0_{s_j}$, and $p^t_{s_k} = p^0_{s_k} \in [0,1]$ for all $s_k \in S$ for all $t \in \mathbb{N}$. Let for $\mathbf{p}^0_{-S}$, $\mathbf{p}^0_S$, $\mathbf{T}^0$, $c \geq 0$, and $a \geq 1$, $\{N_1, \ldots, N_K\}$ be the resulting partitioning in cliques described in Theorem 2. Then for a clique N_k that has more than one persistent member, define $S_k = N_k \cap S \neq \emptyset$. Then $\lim_{t\to\infty} p^t_i = p_\beta$ for all non-persistent members $i \in N_k \setminus S$, and $\lim_{t\to\infty} \mathbf{T}^t = \mathbf{T}^*$ satisfies the following:*

$$\sum_{s\in S_x} (\mathbf{T}^*_{\cdot s} p^0_s) = p_\beta \sum_{s\in S_x} \mathbf{T}^*_{\cdot s}, \tag{19}$$

$$\frac{\mathbf{T}^*_{\cdot s_i}}{\mathbf{T}^*_{\cdot s_j}} = \frac{x\left[(a+2)p^0_{s_i} - 1\right] + |N_k|\left(1 - p^0_{s_i}\right)}{x\left[(a+2)p^0_{s_j} - 1\right] + |N_k|\left(1 - p^0_{s_j}\right)}, \text{ for all } s_i, s_j \in S_x, \tag{20}$$

where $x = \sum_{i\in N_k} p^*_i = (|N_k| - |S_k|)p_\beta + \sum_{s\in S_x} p^0_s$.

When we have a component with one or no diverse persistent players, the final outcomes are the same as described in case of the basic model and uniform persistent players. Theorem 4 follows immediately from the French-DeGroot updating rule demonstrated in (10) and the proof is omitted. Since the final strategy for non-persistent players stated in the theorem is a convex combination of the persistent players in the component, we also know that $\underline{p}_{S_x} \leq p_\beta \leq \overline{p}_{S_x}$, where $\underline{p}_{S_x} = \min_{S_x} p^0_s$ and $\overline{p}_{S_x} = \max_{s\in S_x} p^0_s$.

For low linking costs, we arrive at an analogue of Corollaries 3.5 and 4.1, which states that the updating process converges to a fully connected group with a consensus that is a convex combination of the endowed strategies of the persistent players in the population:

Corollary 4.2 *Consider a situation in which $c < \underline{\pi}$ and the subset of persistent players, the social learning process converges to a fully connected interaction network, identical weights, and all non-persistent players' strategies converge to some $p_\beta \in [0,1]$.*

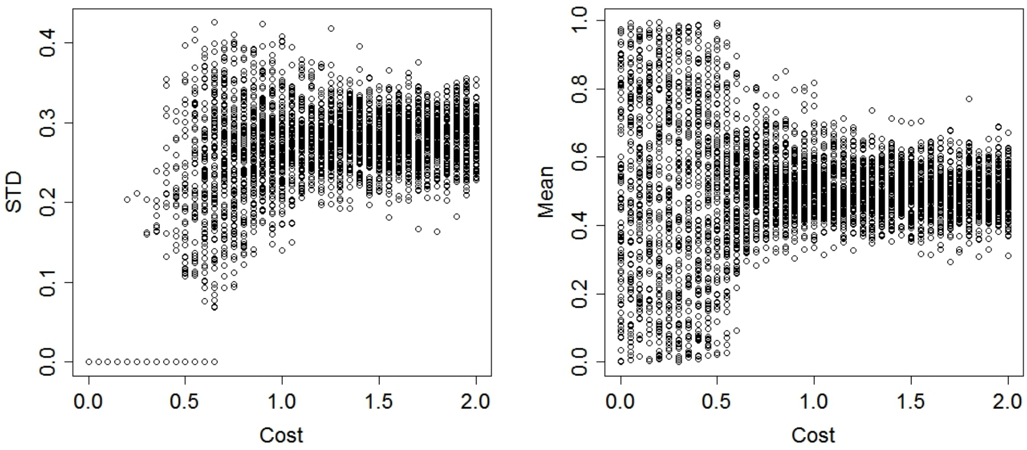
Obviously, when we have persistent players and high cost, the outcomes are the same as those without persistent players. Namely, all players stay isolated and keep their initial strategies in the steady state.

Corollary 4.3 *Consider a situation with persistent players. If $c \geq \overline{\pi}$, then the updating process converges to a situation in which there is an autarkic interaction network, the influence matrix is equal to the identity matrix, and all players choose their initial strategy p^0_i.*

The proof of the corollaries in this section are rather straightforward and therefore omitted.

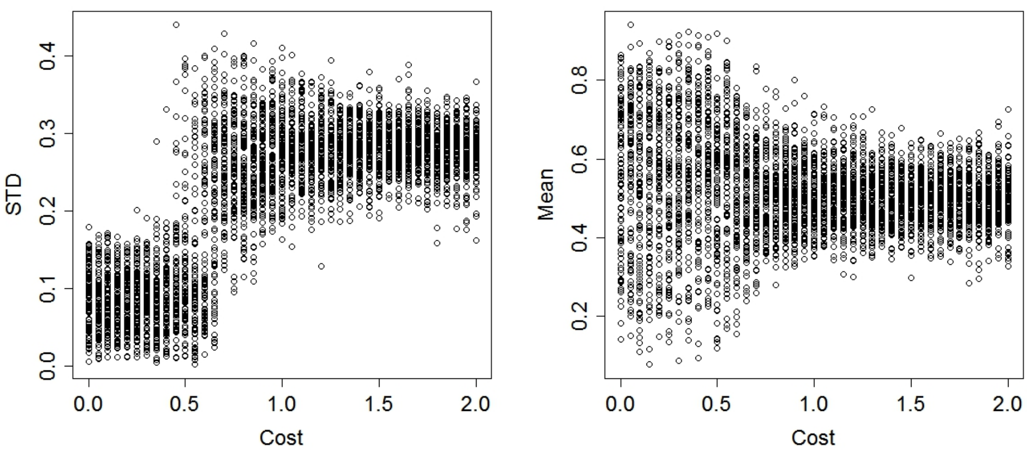
Sensitivity analysis

We use the same settings as for the basic model, where $a = 2$ and c ranges from 0 to 2 in increments of 0.05. At each cost level, 100 simulations are executed.



(a) Standard deviation (b) Mean value

Figure 3: Cost sensitivity analysis with a single persistent player



(a) Standard deviation (b) Mean value

Figure 4: Cost sensitivity analysis with 3 diverse persistent players

Similar to the basic model, society size does not affect the final outcomes. Thus, for both uniform and diverse persistent players, we only show the cases where $n = 20$ with the x-axis showing value of cost and y-axis showing standard deviation and mean value of the final strategies. Figure 3 illustrates the case where we have a single persistent player which represents the uniform persistent model, which shows very similar patterns as the observations in Figure 2. When the standard deviation is 0 in this case, we know the final strategy equals the initial strategy that the uniform persistent player insists on.

In Figure 4 we have 3 diverse persistent players, assigned 3 different initial strategies. Simulation results show that learning with different number of diverse persistent strategies exhibits similar outcome patterns. However, the standard deviation is never 0 when we have diverse persistent players, whose strategies are never equal.

At a first glance, it appears that the mean value plot in the previous section shows a more uniform distribution. Whereas when persistent players are introduced, the mean value falls into a more narrow range when cost is medium to high. If we take a closer look, we see that the range of the mean value in Figure 2 is roughly between 0.4 and 0.6, which is the same as that in Figure 3 and 4 at a higher cost. When cost is lower in case of persistent players, the final strategies are greatly determined by persistent player's initial strategies, which follows a uniform distribution between 0 and 1. And that is the reason why the plot seems to be spreading wider at a lower cost in Figure 3 and 4.

5 Further research questions

The social learning model discussed in this paper has a dynamic double-layer network structure. Namely, players play a coordination game with selected partners in an interaction network; on the other hand, they collect information about the resulting payoffs and the executed strategies in an influence network. Subsequently they update their interaction network, their influence weights, as well as the strategy according to a naïve social learning process.

It is a novel idea to separate the interaction network from the information collection process, taking account of the consideration that individuals first tend to collect information and then process the information, before making a decision on a task or activity that is affiliated with her chosen partners. In our framework both networks are endogenous and change over time. There is also a clear correlation between the two networks that ties them together in a sensible way.

We show that the learning process converges to independent, fully connected cliques within each of which full (local) consensus is reached. Furthermore, if the interaction cost is low enough, the population of all players conform to a global consensus. The steady state is rarely a Nash equilibrium which is only achieved with extreme initial conditions.

Persistency in behavior has an extraordinarily large effect on the steady state conventions that emerge in the system. It shows that stubbornness pays off and pulls the local convention closer to the promoted position or strategy. We stress that with the endogenous network formation in our model, the influence of persistent players is limited comparing to models with similar settings and guaranteed connectivity, where such influence is global.

These insights direct us to some obvious follow-up questions for further research. First, how do certain features in the network architecture affect convergence speed? Preliminary simulations show noticeable effects on convergence speed, but such relationship needs more careful investigation as there seems to be a correlation to initial conditions and details of updating, especially with more complex network structures such as scale-free and small-world.

Second, our framework seems to be a stepping stone for development of multi-issue interaction models that take account of higher-dimensional strategies and more complex information structures.

Finally, our framework is founded on extending the naïve social learning process of French (1956), Harary (1959) and DeGroot (1974) to a setting of value-generating interactions. Similarly, one could aim to extend evolutionary game-theoretic frameworks to include opinion formation and information processing. This requires the development of such an evolutionary theory of information dissemination in an opinion formation setting.

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Appendix

Proof of Proposition 3.2

Denote for each t,

$$\underline{p}^t = \min\{p_1^t, \dots, p_n^t\}, \quad \overline{p}^t = \max\{p_1^t, \dots, p_n^t\}.$$

Thus $p_i^t \in [\underline{p}^t, \overline{p}^t]$ for all i. Moreover, $p_i^{t+1} \in [\underline{p}^t, \overline{p}^t]$ for all i since p_i^{t+1} is a convex combination of p_i^t. Consequently, we have $\underline{p}^t \le \underline{p}^{t+1} \le \overline{p}^{t+1} \le \overline{p}^t$ for all $t \in \mathbb{N}$. Therefore, the proof below applies to all assertions of the proposition since it is valid for all cases where we have $\underline{p}^t \le x, y \le \overline{p}^t$.

Recall that $\pi_{ij}^t = ap_i^t p_j^t + (1 - p_i^t)(1 - p_j^t)$. We have $\frac{\partial \pi_{ij}^t}{\partial p_j^t} = (a+1)p_i^t - 1$. So π_{ij}^t increases with p_j^t when $p_i^t \ge \frac{1}{a+1}$ and decreases with p_j^t when $p_i^t < \frac{1}{a+1}$.

1. $\frac{1}{a+1} \le \underline{p}^t \le \overline{p}^t$.
   In this case, define $\pi_{min}^1 = a(\underline{p}^t)^2 + (1 - \underline{p}^t)^2$, $\pi_{max}^1 = a(\overline{p}^t)^2 + (1 - \overline{p}^t)^2$. Then for all i, t, $p_i^t \ge \underline{p} \ge \frac{1}{a+1}$, $\frac{\partial \pi_{ij}^t}{\partial p_i^t} \ge 0$. Thus, $\pi_{ij}^t \ge \pi_{min}^1$, for all i, j, t. Similarly, $\pi_{ij}^t \le \pi_{max}^1$, for all i, j, t

2. $\underline{p}^t \le \overline{p}^t < \frac{1}{a+1}$.
   In this case, define $\pi_{min}^2 = a(\overline{p}^t)^2 + (1 - \overline{p}^t)^2$, $\pi_{max}^2 = a(\underline{p}^t)^2 + (1 - \underline{p}^t)^2$. Then for all i, t, $p_i^t \le \overline{p} < \frac{1}{a+1}$, $\frac{\partial \pi_{ij}^t}{\partial p_i^t} < 0$. Thus, $\pi_{ij}^t \ge \pi_{min}^2, \pi_{ij}^t \le \pi_{max}^2$, for all i, j, t.

3. $\underline{p}^t < \frac{1}{a+1} \le \overline{p}^t$.
   In this case, define $\pi_{min}^3 = a\underline{p}^t\overline{p}^t + (1 - \underline{p}^t)(1 - \overline{p}^t)$. For arbitrary i, j, t, without loss of generality, assume that $p_i^t \le p_j^t$. We have 2 possible scenarios in this case.

   (i) $p_i^t < \frac{1}{a+1} \le \overline{p}^t$. Then since $\frac{\partial \pi_{ij}^t}{\partial p_j^t} < 0$, $\overline{p}^t \ge p_j^t$, we have $\pi_{ij}^t \ge ap_i^t\overline{p}^t + (1 - p_i^t)(1 - \overline{p}^t)$. And $\frac{\partial \pi_{ij}^t}{p_i^t} > 0$, $\underline{p}^t \le p_i^t$, so

   $$ap_i^t\overline{p}^t + (1 - p_i^t)(1 - \overline{p}^t) \ge a\underline{p}^t\overline{p}^t + (1 - \underline{p}^t)(1 - \overline{p}^t) = \pi_{min}^3,$$

   which implies that $\pi_{ij}^t \ge \pi_{min}^3$.
   Similarly, $\pi_{ij}^t \ge ap_i^t\underline{p}^t + (1 - p_i^t)(1 - \underline{p}^t) \ge a(\underline{p}^t)^2 + (1 - \underline{p}^t)^2 = \pi_{max}^2$

   (ii) $\frac{1}{a+1} \le p_i^t \le \overline{p}^t$. Then similar to the previous case,

   $$\pi_{ij}^t \ge ap_i^t\underline{p}^t + (1 - p_i^t)(1 - \underline{p}^t) \ge a\underline{p}^t\overline{p}^t + (1 - \underline{p}^t)(1 - \overline{p}^t) = \pi_{min}^3,$$

   $$\pi_{ij}^t \le ap_i^t\overline{p}^t + (1 - p_i^t)(1 - \overline{p}^t) \le a(\overline{p}^t)^2 + (1 - \overline{p}^t)^2 = \pi_{max}^1.$$

That is, in all cases, we can find a π_{min}^k such that $\pi_{ij}^t \ge \pi_{min}^k$ for all i, j, t. Define $\underline{\pi}^t = \min\{\pi_{min}^1, \pi_{min}^2, \pi_{min}^3, \}$, then it holds that $\pi_{ij}^t \ge \underline{\pi}^t$ for all i, j, t in all cases. Since $0 \le \underline{p}^t \le \overline{p}^t \le 1$, $a\underline{p}^t\overline{p}^t + (1 - \underline{p}^t)(1 - \overline{p}^t) \ge 0$. Also, the other 2 π_{min} take the form of $a\rho^2 + (1-\rho)^2 \ge \frac{a}{a+1} > 0$ for $0 \le \rho \le 1$. Thus $\underline{\pi}^t \ge 0$; the equality holds only when $\underline{p}^t = 0$ and $\overline{p}^t = 1$.

Likewise, in all cases, we can find a π_{max} such that $\pi^t_{ij} \geq \pi_{max}$ for all i, j, t. We have 2 candidates for π_{max}, namely π^1_{max} and π^2_{max}. Therefore, define $\overline{\pi} = \max\{\pi^1_{max}, \pi^2_{max}\} > \frac{a}{a+1} > 0$, we have $\pi^t_{ij} \leq \pi_{max}$ for all i, j, t.

Proof of Corollary 3.3

In this proof for all statements involving t, it is implied that t is sufficiently large such that M is a component in $\mathbf{G}^t$.

First, since M is a limit component, $\mathbf{G}^t_{ij} = 0$ for all $i \in M$, $j \notin M$. Then we know from the influence weight updating rule that $\mathbf{T}^t_{ij} = 0$ for such i, j. Since $\mathbf{T}^t$ is always stochastic, in turn we have $\sum_{j \in M} \mathbf{T}^t_{ij} = 1$ for all $i \in M$. In other words, for all $i \in M$, p^t_i is a convex combination of $\{p^t_j \mid j \in M\}$ only. Therefore, for all $i \in M$, $p^{t+1}_i \in \left[\underline{p}^t_M, \overline{p}^t_M\right]$. This further implies that $\underline{p}^{t+1}_M \geq \underline{p}^t_M$ and $\overline{p}^{t+1}_M \leq \overline{p}^t_M$.

Next, define $\overline{M}^t = \left\{i \in M \mid p^t_i = \overline{p}^t_M\right\}$ and $\underline{M}^t = \left\{i \in M \mid p^t_i = \underline{p}^t_M\right\}$. Since M is a component in $\mathbf{G}^t$, there exists $i \in \overline{M}^t$ and $j \in M \setminus \overline{M}^t$ such that $\mathbf{G}^t_{ij} = 1$. Consequently, $\mathbf{T}^t_{ij} > 0$. Since $p^t_j < p^t_i$, taking the convex combination results in $p^{t+1}_i < p^t_i = \overline{p}^t_M$. We can repeat this process $\overline{k}$ times, where $1 \leq \overline{k} \leq |\overline{M}^t|$, after which $p^{t+\overline{k}}_i < p^t_i = \overline{p}^t_M$ for all $i \in \overline{M}^t$. Therefore, $\overline{p}^{t+\overline{k}}_M < \overline{p}^t_M$.

Similarly, there exists $1 \leq \underline{k} \leq |\underline{M}^t|$ such that $\underline{p}^{t+\underline{k}}_M < \underline{p}^t_M$. Now we can simply mimic the proof of Proposition 3.2 given above to show that the assertion in the lemma is true.

Proof of Proposition 3.4

First, the conditions stated in the assertion indicate that the elements of $\mathbf{p}^0$ cannot be all 0s or all 1s, in which case we won't have $i, j \in N$ such that $p^0_i \neq p^0_j$.

As shown in the proof of Proposition 3.2, $\{\mathbf{p}^t\}^\infty_{t=0}$ is a sequence in a compact set $[\underline{p}^0, \overline{p}^0]^n$. So if $0 < p^0_i < 1$ for all i, then the assertion of Proposition 3.4 is true.

Next, consider the case where there exists $\gamma \in N$ such that p^0_γ is either 0 or 1. Then in order to have $p^t_\gamma = p^0_\gamma$, it must hold that $\mathbf{T}^t_{\gamma j} > 0$ if $p^0_\gamma = p^0_j$ and $\mathbf{T}^t_{\gamma j} = 0$ otherwise. Suppose that $\mathbf{T}^0$ satisfies that condition. Recall that

$$\mathbf{T}^t_{ij} = \frac{w^t_{ij}}{\sum^n_{k=1} w^t_{ik}}, \text{ for all } j \in N, t > 0, \quad \text{where } w^t_{ij} = \sum_{l \in L^t_i} \mathbf{G}^t_{lj} \pi^{t-1}_{lj}.$$

Consider two players γ, j such that $p^0_\gamma \neq p^0_j$. Note that when $c < \underline{\pi}^0$, a link between any pair of players is formed with probability 1. So there exists $\hat{t}_\gamma$, s.t. for $t > \hat{t}_\gamma$, $j \in L(\gamma)^t$ and $j \in L(j)^t$. $\pi^t_{jj} = k(p^t_j)^2 + (1 - p^t_j)^2 > 0$, which means that $w^t_{\gamma j} > 0$. Thus, $\mathbf{T}^t_{\gamma j} > 0$ even though $p^0_\gamma \neq p^0_j$. In other words, player γ cannot remain her initial strategy of 0 or 1. We can repeat this process for all $\{\lambda \mid \lambda \in N, p^0_\lambda = 0 \text{ or } 1\}$. Thus, defining $\hat{t} = \max_\lambda \hat{t}_\lambda$ it holds that for every $t > \hat{t}$ and $k \in N$: $0 < p^t_k < 1$.

Proof of Theorem 1

Before addressing the proof of Theorem 1, we develop a number of intermediary results.

Lemma 1 *$\mathbf{T}^t_{ii}$ has a lower bound δ, i.e., $\mathbf{T}^t_{ii} \geq \delta$ for all i, for all t, for all $\mathbf{T}^0$, $\mathbf{p}^0$, $c \geq 0$ and $a \geq 1$.*

Proof. Since $\mathbf{G}^t_{ii} = 1$, or equivalently, $i \in L^t_i$ for all i for all t, with (9) we have for $0 \leq p^{t-1}_i \leq 1$ that

$$w^t_{ii} = \sum_{l \in L^t_i} \mathbf{G}^t_{li}\pi^{t-1}_{li} \geq \pi^{t-1}_{ii} = a(p^{t-1}_i)^2 + (1 - p^{t-1}_i)^2 \geq \frac{a}{a+1}$$

And consequently,

$$\mathbf{T}^t_{ii} = \frac{w^t_{ii}}{\sum_{k=1}^n w^t_{ik}} \geq \frac{\frac{a}{a+1}}{\sum_{k=1}^n w^t_{ik}} \geq \frac{\frac{a}{a+1}}{na} = \frac{1}{n(a+1)}$$

where the second inequality follows from the property that $\pi^t_{ij} \leq a$ for all i, j, and t. Define $\delta = \frac{1}{n(a+1)}$, then $\mathbf{T}^t_{ii} \geq \delta \geq 0$ for all i, for all t, for all $\mathbf{T}^0$, $\mathbf{p}^0$, $c \geq 0$ and $a \geq 1$. ■

Lemma 2 *$\mathbf{T}^t_{ij} = 0$ if and only if $\mathbf{T}^t_{ji} = 0$ for all i, j, t.*

Proof. Recall that $w^t_{ij} = \sum_{l \in L^t_i} \mathbf{G}^t_{lj}\pi^{t-1}_{lj}$. We see that $w^t_{ij} = 0$ if and only if $\mathbf{G}^t_{lj}\pi^{t-1}_{lj} = 0$ for all $l \in L^t_i$. With the interaction network updating rule (7), $\mathbf{G}^t_{lj} = 1$ if and only if $\pi^{t-1}_{lj} > c$. Since $c \geq 0$, $\pi^{t-1}_{ij} > 0$. That is, $\mathbf{G}^t_{lj}\pi^{t-1}_{lj} = \pi^{t-1}_{lj} > 0$ if $\mathbf{G}^t_{lj} = 1$.

Therefore, $w^t_{ij} = 0$ if and only if $\mathbf{G}^t_{lj} = 0$ for all $l \in L^t_i$. That is, i and j is not directly connected (since $i \in L^t_i$) or connected through a middle player. Since the interaction network $\mathbf{G}^t$ is symmetric, this implies that $\mathbf{G}^t_{li} = 0$ for all $l \in L^t_j$, which means that $w^t_{ji} = 0$.

In conclusion, $w^t_{ij} = 0$ if and only if $w^t_{ji} = 0$. Then we have $\mathbf{T}^t_{ij} = \dfrac{w^t_{ij}}{\sum_{k=1}^n w^t_{ik}} = 0$ if and only if $\mathbf{T}^t_{ji} = \dfrac{w^t_{ji}}{\sum_{k=1}^n w^t_{jk}} = 0$ for all i, j, t. ■

Lemma 3 *There is a lower bound for positive $\mathbf{T}^t_{ij}$, i.e., $\mathbf{T}^t_{ij} \geq \hat{\delta}$ if $\mathbf{T}^t_{ij} > 0$ for all i, for all t, for all $\mathbf{T}^0$, $\mathbf{p}^0$, $c \geq 0$ and $a \geq 1$.*

Proof. First, $\mathbf{T}^t_{ij} > 0$ if and only if $w^t_{ij} > 0$. From the proof for Lemma 2, we know that $w^t_{ij} > 0$ if and only if there exists l, such that $\mathbf{G}^t_{lj} = 1$ and $\pi^{t-1}_{lj} > c$. Therefore $w^t_{ij} = \sum_{l \in L^t_i} \mathbf{G}^t_{lj}\pi^{t-1}_{lj} > c$ for all i, j, t such that $w^t_{ij} > 0$.

Then we have $\mathbf{T}^t_{ij} = \dfrac{w^t_{ij}}{\sum_{k=1}^n w^t_{ik}} > \dfrac{c}{na}$ for all i, j, t such that $\mathbf{T}^t_{ij} > 0$. Denote $\hat{\delta} = \dfrac{c}{na}$, then we have $\mathbf{T}^t_{ij} > \hat{\delta}$ if $\mathbf{T}^t_{ij} > 0$. ■

Now, to prove Theorem 1, we use Theorem 2 in Lorenz (2005). To use it, $\mathbf{T}^t$ needs to satisfy the following conditions for all t.

C1 $\mathbf{T}^t_{ii} > 0$ for all i.

C2 $\mathbf{T}^t_{ij} = 0$ if and only if $\mathbf{T}^t_{ji} = 0$ for all i, j.

C3 There exist $\hat{\delta} > 0$, such that $\mathbf{T}^t_{ij} > \hat{\delta}$ for all i, j such that $\mathbf{T}^t_{ij} > 0$.

Lemma 1, 2 and 3 show that all the 3 conditions are satisfied. Then by Theorem 2 in Lorenz (2005), $\mathbf{T}^t$ is convergent, and with simultaneous row and column permutations, the limit of the product of the influence matrices takes the following form:

$$\lim_{t\to\infty}\prod_{\tau=t}^{1}\mathbf{T}^t = \mathbf{T}^* = \begin{pmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_k \end{pmatrix},$$

where $T_1, \ldots, T_k$ are stochastic matrices with identical rows. Such matrices are called "consensus matrices" because the weighted averages with these matrices leads to consensus within a group of individuals corresponding to the indices of each matrix.

That is, we can have a partition $\{N_1, \ldots, N_k\}$ corresponding to the indices of the consensus matrices. Then for a subset N_x, players in the set converge to have the same strategy p_k in the steady state.

Now that we have covered the local convention, we turn to check the connectivity among players in N_x. First, note that the zero pattern indicates no connection between players in 2 different partition subsets. Second, note that N_x can be singular (has only 1 player), in which case the fully connectedness is trivial. Then, we look at the cases where N_x has at least 2 players.

Note that from the convergence of p_i^t it follows immediately that

$$\lim_{t\to\infty}\pi_{ij}^t = a(p_x)^2 + (1-p_x)^2 \equiv \pi_x, \qquad \text{for all } i, j \in N_x.$$

Next, we show that players in N_x cannot be isolated with large enough t. Suppose for any $t_x \in \mathbb{N}$, there exists $i \in N_x$ and $t > t_x$ such that $\mathbf{T}_{ii}^t = 1$ and $\mathbf{T}_{ij}^t = 0$ for all $j \neq i$. Take the product of a matrix with the i-th row in this form preserves the pattern, i.e., $\mathbf{T}_{ii}^{t+1} = 1$ and $\mathbf{T}_{ij}^{t+1} = 0$ for all $j \neq i$. This is a contradiction to the assertion that T_x converges to a block in the matrix with positive and identical rows.

Hence, there exist $\hat{t}_x$ such that for all $t > \hat{t}_x$, we have $i, j \in N_x$ such that $\mathbf{G}_{ij}^t = 1$. Since the link requires $\pi_{ij}^t > c$, we have $\pi_x > c$. Consequently, a link between any pair of players in N_x is formed and maintained with probability 1, i.e., $\mathbf{G}^t$ on N_x converges to be fully connected.

For the same reason, if there exists $j \notin N_x$ such that $\lim_{t\to\infty} p_j^t = p_x$, $\pi_{ij}^t \to \pi_x > c$ for any $i \in N_x$, which implies that j cannot be out of the component N_x.

The proof for the last assertion regarding to the convergence pattern of influence weights is straightforward. Given that N_x is fully connected, for all $i, j \in N_x$ we have that

$$w_{ij}^t = \sum_{k\in N} G_{kj}^{t-1}\pi_{kj}^t = \sum_{k\in N_x} G_{kj}^{t-1}\pi_{kj}^t \to \sum_{k\in N_x} 1\cdot\hat{\pi} = |N_x|\cdot\hat{\pi} \equiv \hat{w}.$$

This in turn implies that for $i, j \in N_x$, $T_{ij}^t \to \dfrac{\hat{w}}{\sum_{k\in N_x}\hat{w}} = \dfrac{1}{|N_x|}$, showing the desired property.

Proof of Theorem 2

Note that we use $\tilde{\mathbf{T}}^t$ defined in (18) in this case. Since we have persistent players who essentially place weight 1 on themselves and 0 on everybody else, Lemma 3 is no longer true. Namely, suppose a non-persistent i has a connection with a persistent player s, and

$\pi^t_{is} > 0$. Then $\tilde{\mathbf{T}}^t_{is} = \mathbf{T}^t_{is} > 0$ because $w^t_{is} \geq \pi^t_{is} > 0$; whereas $\tilde{\mathbf{T}}^t_{si} = 0$. Since the symmetric zero weight condition is not satisfied, we cannot use Theorem 2 in Lorenz (2005) as we did before.

Instead, we use Proposition 3 in Lorenz (2005) which requires only the validity of positive diagonal elements (C1). In this case, Lemma 1 is still applicable to non-persistent players, i.e., $\tilde{\mathbf{T}}^t_{ii} = \mathbf{T}^t_{ii} > 0$ for all $i \notin S$. And we know that $\tilde{\mathbf{T}}^t_{ss} = 1 > 0$ for all $s \in S$. Thus $\tilde{\mathbf{T}}^t$ has positive diagonal elements for all t.

Then Lorenz's Proposition 3 states that there exists a series of time steps $t_0 < t_1 < \ldots$ such that $\prod_{\tau=t_i}^{t_{i-1}} \tilde{\mathbf{T}}^t$ has the same zero patterns for all t_i. Define $\tilde{\mathbf{T}}^{t_i} = \prod_{\tau=t_i}^{t_{i-1}} \tilde{\mathbf{T}}^t$, the updating rule can be rewritten as:

$$\mathbf{p}^{t_i} = \tilde{\mathbf{T}}^{t_i}\mathbf{p}^{t_{i-1}}. \tag{21}$$

Next we construct a partition of N: $\{N_1, \ldots, N_k\}$ such that players in each group N_x place positive weights on each other, i.e., $\tilde{\mathbf{T}}^{t_i}_{x_i x_j} > 0$ for all $x_i, x_j \in N_x$ for all t_i. Since the zero patterns are identical for all $\tilde{\mathbf{T}}^{t_i}$, the partition is also fixed.

A persistent player is always in a singular group since she never places positive weight on others. However, a non-persistent player (in a different group) may place positive weights on her if they have a connection in $\mathbf{G}^t$. And the positive weights carry on through matrix multiplication as well when the matrix always has positive diagonal. That is, with this partition, a player may place positive weight on another who is not in the same group.

Using one of the variations of Theorem 3.1 in Seneta (1981), for $| N_x | \geq 2$ (which implies that all players in the set are non-persistent), we have

$$\max_{i,j \in N_x} |p^{t_{i+1}}_i - p^{t_i+1}_j| \leq \mu_{t_i}(\tilde{\mathbf{T}}_{N_x})\{\max_{i,j \in N_x} |p^{t_i}_i - p^{t_i}_j|\}, \tag{22}$$

where

$$\mu_{t_i}(\tilde{\mathbf{T}}_{N_x}) = \frac{1}{2} \max_{i,j \in N_x} \sum_{k=1}^{n} |\tilde{\mathbf{T}}^{t_i}_{ik} - \tilde{\mathbf{T}}^{t_i}_{jk}|. \tag{23}$$

For $i, j \in N_x$, define $D^{ijt}_+ = \{k \mid \tilde{\mathbf{T}}^t_{ik} \geq \tilde{\mathbf{T}}^t_{jk}\}$ and $D^{ijt}_- = \{k \mid \tilde{\mathbf{T}}^t_{ik} \leq \tilde{\mathbf{T}}^t_{jk}\}$. Since $\sum_{k=1}^{n} \tilde{\mathbf{T}}^t_{ik} = 1$, and $\sum_{k=1}^{n} \tilde{\mathbf{T}}^t_{jk} = 1$ for all i, j, t, neither set is empty. Besides,

$$\sum_{k \in D^{ijt}_+} (\tilde{\mathbf{T}}^t_{ik} - \tilde{\mathbf{T}}^t_{jk}) = \sum_{k \in D^{ijt}_-} (\tilde{\mathbf{T}}^t_{jk} - \tilde{\mathbf{T}}^t_{ik}). \tag{24}$$

Then we have:

$$\mu_{t_i}(\tilde{\mathbf{T}}_{N_x}) = \max_{i,j \in N_x} \left\{ \sum_{k \in D^{ijt_i}_+} \tilde{\mathbf{T}}^{t_i}_{jk} - \sum_{k \in D^{ijt_i}_+} \tilde{\mathbf{T}}^{t_i}_{ik} \right\}. \tag{25}$$

By the rule of matrix multiplication, $\tilde{\mathbf{T}}^{t_i}_{ij} > 0$ if and only if there exists $t' \leq t_i$ such that

$\mathbf{T}^{t'}_{ij} >> 0$. Again with the rule of matrix multiplication, for all $t > t'$, $\mathbf{T}^t_{ij}$ is multiplied by $\mathbf{T}^t_{jj}$, which has a lower bound δ defined in Lemma 1. So $\tilde{\mathbf{T}}^{t_i}_{ij} > (\delta)^{t_i - t'}$ that is also greater than and bounded away from 0, since $t_i - t'$ is bounded.

That is, here we also have a lower bound for $\tilde{\mathbf{T}}^{t_i}_{ij}$, which is positive by definition of the partition. Thus $\mu_{t_i}(\tilde{\mathbf{T}}_{N_x})$ is less than and bounded away from 1. Note that

$$\max_{i,j\in N_x} |p_i^{t_i+1} - p_j^{t_i+1}| \leq \left[\prod_{\tau=\hat{t}}^{t} \mu_\tau(\tilde{\mathbf{T}}_{N_x})\right] \max_{i,j} |p_i^{t_0} - p_j^{t_0}|. \tag{26}$$

And $\prod_{\tau=\hat{t}}^{t} \mu_\tau(\tilde{\mathbf{T}}) \rightarrow 0$ since $\mu_{t_i}(\tilde{\mathbf{T}}_{N_x})$ is less than and bounded away from 1. We have $\max_{i,j\in N_x} |p_i^{t_i+1} - p_j^{t_i+1}| \rightarrow 0$, which implies that players in N_x converge to a local convention p_x.

Next, we merge groups in the partition following the rule that N_x and N_y merge if and only if $\pi_{xy} = ap_x p_y + (1 - p_x)(1 - p_y) > c$. With the proof for Theorem 1, we know that each N_x is connected because all players in the set place positive weights on all others. Then since the merge exhausts all possible connections, each group $N_{x'}$ in the new partition is indeed a component.

This is an important step that explains the connection between non-persistent and persist players who are in separate groups before (which does *not* mean that they do not have a connection in the interaction network). With the same logic in the proof for Theorem 1, $N_{x'}$ is fully connected among non-persistent players. We emphasize that the persistent players in the component could be fully connected as well, which is indeterminable in a general case. And for all $i \in N_{x'}$, $i \notin S$, for all $j \in N$,

$$\lim_{t\to\infty} \mathbf{T}^t_{ij} = \frac{w^*_{j,N_{x'}}}{\sum_{j\in N} w^*_{j,N_{x'}}}, \tag{27}$$

where $w^*_{j,N_{x'}} = \sum_{i\in N_{x'}} ap_i^* p_j^* + (1 - p_i^*)(1 - p_j^*)$. Note that $\lim_{t\to\infty} p_i^t = p_i^* = p_j^* = \lim_{t\to\infty} p_j^t$ for all $i, j \in N_{x'}$, $i, j \notin S$. And $\lim_{t\to\infty} p_s^t = p_s^* = p_s^0$ for all $s \in N_{x'}$, $s \in S$.

That is, the influence weights for all players in a component are identical.

Proof of Theorem 3

By Theorem 2, $\mathbf{G}^t$ is convergent. And with a component N_x defined by $\mathbf{G}^*$, we know that $\lim_{t\to\infty} p_i^t = p^*_{N_x}$ for all $i \in N_x$, $i \notin S$. Also, $\mathbf{T}^t_{N_x}$ converges to have identical row.

Given that there are persistent players in N_x, define $S_x = \{s \mid s \in N_x, s \in S\}$. Immediately following the French-DeGroot updating rule indicated in (10), we have

$$\sum_{s\in S_x} \{\mathbf{T}^*_{\cdot s} p_\alpha\} = \sum_{s\in S_x} \left\{\mathbf{T}^*_{\cdot s} p^*_{N_x}\right\}. \tag{28}$$

Since both p_α and $p^*_{N_x}$ are constants, we have

$$p_\alpha \sum_{s\in S_x} \{\mathbf{T}^*_{\cdot s}\} = p^*_{N_x} \sum_{s\in S_x} \{\mathbf{T}^*_{\cdot s}\}. \tag{29}$$

The directly induction is that $p_\alpha = p^*_{N_x}$, i.e., the non-persistent players converge to the (uniform) persistent players' initial strategy.

Assertion (b) follows the same reasoning used in the proof for Theorem 1 on unique local consensus. Suppose there are 2 persistent players that belong to 2 different limit components N_x and N_y. Then we have $\lim_{t\to\infty} p_i^t = p_\alpha$ for all $i \in N_x \cup N_y$. Consequently, we know that $\lim_{t\to\infty} \pi_{ij}^t = \pi_\alpha$ for all $(i, j) \in N_x \times N_y$. That is, the 2 components merge with probability 1 and all pairs in the merged component form links with probability 1. In other words, all persistent players are in the same limit component if 1 persistent player is not isolated. The only other possibility is that all persistent players are isolated and not connected to anyone.

Then following the same proof for Theorem 1, $\lim\limits_{t\to\infty} \mathbf{T}_{ij}^t = \dfrac{1}{\mid N_x \mid}$ for all $i, j \in N_x$. Since N_x is a component, $\mid N_x \mid = \mid L_i^* \mid$. So $\lim\limits_{t\to\infty} \mathbf{T}_{ij}^t = \dfrac{1}{L_i^*}$ for all i, for all $j \in L_i^*$. Note that this also covers the weight distribution for persistent players, which is not in practical use when updating and different from the weight defined in (18).