Stability in a Network Economy: The Role of Institutions*

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Abstract

We consider an economy in which agents are embedded in a network of potential value-generating relationships. Agents are assumed to be able to participate in three types of economic interactions: Autarkic self-provision; bilateral interaction; and multilateral collaboration.

We introduce two stability concepts and provide sufficient and necessary conditions on the network structure that guarantee existence, in cases of the absence of externalities, link-based externalities and crowding externalities. We show that institutional arrangements based on socioeconomic roles and leadership guarantee stability. In particular, the stability of more complex economic outcomes requires more strict and complex institutional rules to govern economic interactions. We investigate strict social hierarchies, tiered leadership structures and global market places.

Keywords: Network economies; Bilateral and multilateral interaction; Institutions; Institutional stability.

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1 Stability in a network economy

Stability is universally accepted as a desirable feature in economic analysis. In this paper we study institutional arrangements that facilitate the emergence of stable outcomes and, thus, can be seen as promoters of economic development and growth. Instability, on the other hand, is manifested in a dysfunctional institutional organisation of the economy.

For economists, stability implies not only predictability, but also gains in happiness through the reduction of uncertainty and risk (Dehejia, DeLeire, and Luttment, 2007) and is generally seen as being conducive to economic growth. For example, Mobarak (2005) provides empirical evidence on the relation between (in)stability and economic growth. Institutions can be considered to provide stability if they facilitate the emergence of stable outcomes, surviving long-term evolutionary change of preferential tastes and productive abilities and technologies.

For a stable outcome of economic interactions to emerge, our study shows that institutional arrangements based on hierarchical leadership and market making might restrict interactions among agents, but on the other hand facilitate the emergence of a stable state or outcome in the economy and, thus, function as "stabilisers". Institutions identified here as stabilisers in our stylised economy are thus fulfilling a mechanistic role in the evolution of human organisation as discussed by Stoelhorst and Richerson (2013). Our analysis is also in accord with Kaufman's (2003) theory of economic organisation that builds on the writings of John R. Commons. These authors postulate the necessity of institutional rules which prescribe the domains of decision making. Here we provide a formal proof that such rules are both necessary and sufficient to ensure stability.

We also discuss in what sense our analysis implies the stability of these institutions themselves. Indeed, institutional arrangement like social hierarchies and market making persist throughout human history, since these support and promote stability. Our formal theorems provide a mathematical foundation for this.

We also see our work as complementary to studies of the co-evolution and endogeneity of culture as institutional rules and economic activities (Frederking, 2002; Kuran, 2009). In our formal theory, we de-couple the stabilising function of institutional rules from the content of the economic activities. Thus, we not only gain more universal applicability, but we also identify institutional rules which function robustly in a changing environment of economic activities, albeit within the domain of the type of economic outcomes on which we focus here.

Turning to our formal model, we consider an economy consisting of economic agents who are embedded in a network of potential value-generating relationships. The generated gains from interaction are modelled as (hedonic) utility values over the possible economic activities in which these agents can engage as prescribed by the network. The restrictions implied by the network are interpreted as institutional rules that govern the underlying engagement of agents through the prescribed economic activities.

We use straightforward extensions of standard stability notions from matching theory (Roth and Sotomayor, 1990) and network formation theory (Jackson and Wolinsky, 1996) to define stability in two stylised economic systems. First, we consider a *matching economy* that is founded on bilateral interactions only. Subsequently, we extend our setting to include *multilateral* interactions, where individuals can engage with multiple partners. Such multilateral interactions are akin to multi-sided platforms as considered by Hagiu and Wright (2011) and Evans and Schmalensee (2013) in the context of market theory, extending the seminal work by Rochet and Tirole (2003, 2006) and Evans (2003).

We subsequently identify conditions on the network structure underlying the economy that guarantee the existence of stable bilateral and multilateral outcomes, respectively. These conditions clearly point to institutional features of the underlying network of potential relationships as representing the social capital instilled in these networks (Portes, 1998; Dasgupta, 2005). In particular, in the case of bilateral economic outcomes, we identify the stabilising effects of imposing binary socioeconomic roles such that all economic activities are restricted to occur between agents of two distinct roles. In this respect our work is related to that of Jackson and Watts (2008) who discuss the sufficiency of two distinct market roles for the existence of stability in what in our terms is a bilateral economy. We strengthen their results in this context by showing that this condition is both sufficient and necessary.[1] From this viewpoint, institutional functionality is more closely related to a development process based on the deepening of the social division of labour, in the sense of Smith (1776) and his predecessors (Sun, 2012).

Regarding the stability of multilateral economic outcomes, we identify the necessity of the absence of certain cycles in the underlying network which correspond to the implementation of certain social hierarchies in the represented society. Therefore, macroeconomic properties—described by rules of social authority and hierarchy—are not simply aggregates of microeconomic features, but are "emergent" at the level of the social, institutional governance system in the economy. This interpretation is similar to the notion of *emergence* in a macro economy as put forward by Wagner (2012). Indeed, Wagner's contention is that such institutional rules of economic conduct and interaction are *explanatorily irreducible* in the sense that these rules cannot be devolved to the level of the individual decision makers in the economy.

It is worth pointing out that here we depart from other game-theoretic approaches to the study of stability of institutions that take a dynamic (Goyal and Janssen, 1995) or evolutionary (Sugden, 1995) approach. Our work, instead, treats institutional arrangements

[1] These authors also consider an environment with multiple roles and multilateral outcomes. Their setup differs from ours as we take a network approach. In their multilateral context Jackson and Watts (2008) only provide an example of non-existence of equilibrium whereas we can characterise the necessary and sufficient conditions on the network for the existence of a stable outcome.

as facilitators of economic activity: They provide a well-founded environment in which such activities emerge. Here stability is treated as an intrinsic property of the topology of economic opportunity.

We also distinguish our work from the transaction costs literature, e.g., Coase (1937); North and Thomas (1973); Williamson (1975); North (1990); and Greif (2006), where institutions are usually understood as devices that lower market transaction costs. Lower transaction costs in turn result into increased market efficiency and consequently economic growth and development. On the other hand, our approach takes these institutions as fitting specifications of underlying network properties and brings out their functional role as stabilisers of economic activity.

Furthermore, our work is also related to the literature on theories of economic interaction on network structures as developed by Kranton and Minehart (2000) and Bramoullé and Kranton (2007). These studies develop a purely non-cooperative game theoretic perspective and seek to characterise the resulting equilibrium states. Instead, our study follows the cooperative game theory tradition, being more normative in nature and seeking to answer the question under which institutional configurations stability is guaranteed. Our approach also differs in the nature of the stability notion: Non-cooperative game theory approaches employ Nash-like equilibrium concepts. Here, we adopt a notion of stability which describes cooperative structures that emerge through collaborative formation processes. We view our analysis as complementary to that conducted in this literature.

To show the principles of our approach, we next present some simple examples and debate the concepts that are required to describe the endogenous emergence of stable economic outcomes.

1.1 A motivating example: A hunter-gatherer economy

In this paper we investigate the institutional conditions for *universal stability* in the sense that for *every* pattern of generated consumptive and productive values there exists a stable outcome. We model these institutions as rules that determine the feasible interaction between economic agents.

To illustrate this we consider here the most primitive economic environment, namely that of a hunter-gatherer society. The institutional foundation of all economic interaction is based on the clear separation of individuals through two *socio-economic roles* namely that of a *hunter* and a *gatherer*. The anthropological and economic literature recognises that the social division of labour based on these hunter-gatherer roles has been at the foundation for the initial success and survival of the human species. This research also indicates that it is correct to think of hunters as male and gatherers as female. We refer to Arnold (1993), Flores (1998), Kuhn and Stiner (2006) and Nakahashi and Feldman (2014) for more elaborate a discussion of this.

For ease of the discourse, a bilateral interaction between one hunter and one gatherer is thought of as representing a "household". The institutional framework of a hunter-gatherer society thus results in networks of feasible bilateral interactions, or households.[2] The benefits that individual agents potentially receive from interacting are summarised through hedonic utilities defined over all feasible households.

We assume throughout that every agent can participate in *at most one* household. A bilateral interaction pattern is *stable* if (i) there is no agent who prefers to remain in autarky rather than form the bilateral interaction in the proposed pattern ("individual rationality"); and (ii) there is no hunter-gatherer pair who can and prefer to form that interaction rather than conform to the assigned pattern ("pairwise stability"). We claim now that in a hunter-gatherer economy, regardless of the values generated in these activities, there always exists a stable bilateral interaction outcome.

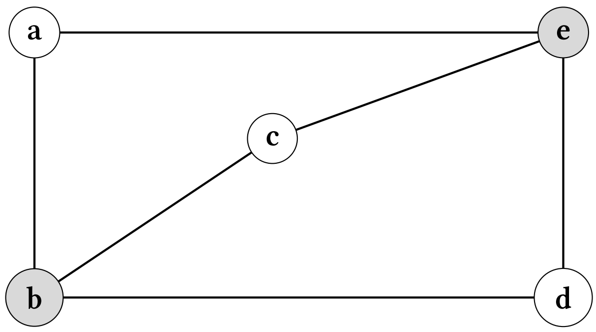

Figure 1: Network structure $\mathcal{A}$

In Figure 1 we consider an example of a society of 5 individuals $N = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$, who have assumed the role of hunter and gatherer. In particular, we let individuals **a**, **c** and **d** be hunters—represented by the white nodes—while **b** and **e** are assumed to be gatherers—represented by the gray nodes. Network $\mathcal{A}$ now consists exactly of all feasible bilateral interactions represented as links between hunters and gatherers.

To focus on the issue of stability, we abstract away from the actual content of the economic interaction itself and use, instead, hedonic values directly. Note that these hedonic utilities are determined also by the socio-economic roles of the agents considered. Hence, these utilities reflect that a hunter and a gatherer are specialised in different productive tasks and that the outputs of these activities are shared in the resulting bilateral interaction. The notion of hedonic games in the context of coalition formation was seminally introduced by Drèze and Greenberg (1980) and further studied by Bogomolnaia and Jackson (2004), Banerjee, Konishi, and Sonmez (2001), and Pápai (2004), among others.[3] We

[2]Similarly, in network games agents can only generate economic surplus if they have a link in the network structure, cf. Kranton and Minehart (2000).

[3]One can also treat these hedonic utilities as being generated by an exchange of goods (Howitt and Clower, 2000), gifts (Akerlof, 1982), favours (Neilson, 1999), or collaborative interactions in clubs (Scotchmer, 2002).

point out that what distinguishes our work from those contributions studying coalition formation games is that we employ a network approach.

For network structure $\mathcal{A}$, we claim that for any pattern of hedonic utility values generated in the identified bilateral interactions there exists at least one stable interaction pattern. For example, let the generated hedonic utilities be given in Table 1.[4]

	a	**b**	**c**	**d**	**e**
a	0	2	–	–	1
b	1	0	2	2	–
c	–	1	0	–	2
d	–	1	–	0	2
e	3	–	1	1	0

Table 1: Hedonic payoffs in Network Structure $\mathcal{A}$.

For agent **a**, $u_a(\mathbf{ae})$ now represents the hedonic utility that hunter **a** gets from setting up a household with gatherer **e**. For the values given in this example there emerge two stable outcomes: $\pi_1 = \{\mathbf{ae}, \mathbf{bc}, \mathbf{dd}\}$ and $\pi_2 = \{\mathbf{ae}, \mathbf{bd}, \mathbf{cc}\}$. In fact, our main result Theorem 3.6 implies that for *every* distribution of hedonic values in network structure $\mathcal{A}$ there exists a stable household pattern.

An institutional cause of instability. Next consider an alternative institutional structure on the population $N = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$. Besides the roles of hunter and gatherer, we now introduce a third socioeconomic role, namely a *chieftain*. The chieftain acts as a communal leader, who has potential bilateral relationships with all other members of the society.[5] A network structure that satisfies this institutional framework is depicted in Figure 2, where **c** is a chieftain; **a** and **d** are hunters; and **b** and **e** are gatherers. Two potential households **ab** and **de** are now supplemented with all hunter-gatherers having a bilateral value-generating interaction with the chieftain **c**.

Network structure $\mathcal{B}$ violates the binary institutional hunter-gatherer framework: One cannot assign hunter-gatherer roles to these 5 agents without violating the institutional rule that only households between agents in different roles can be formed. Indeed, in Figure 2, agent **c** is interacting with all other agents, while also interactions **ab** and **de** exist. This prevents the proper assignment of two roles to $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ which justifies these interactions as households.

[4]This matrix actually represents the *incidence matrix* of network structure $\mathcal{A}$ in which potential payoffs are reported instead of an indicator of connectedness. The number reported in field (i, j) is $u_i(ij)$. Similarly, the field (j, i) reports $u_j(ij)$. If no relationship can be formed, no payoff is reported, indicated by "–". Note that we have normalised the hedonic value of autarky to be 0, i.e. for all $i \in N$, $u_i(ii) = 0$.

[5]The emergence of such leadership in primitive hunter-gatherer societies has been investigated in Redmond and Spencer (2012) and Baker, Bulte, and Weisdorf (2010).

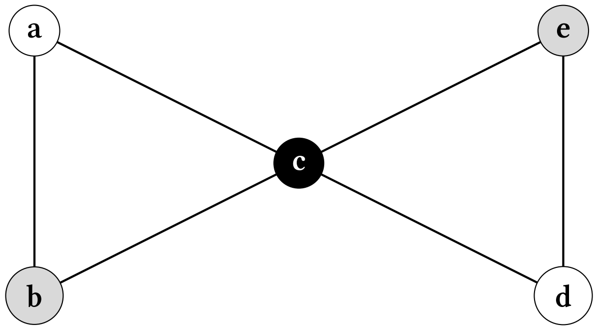

Figure 2: Network structure $\mathcal{B}$

We claim that in network structure $\mathcal{B}$ there is *at least one* profile of hedonic utilities for which there does *not* exist a stable bilateral interaction pattern. Indeed, consider the hedonic utility values of all potential interactions in $\mathcal{B}$ given in Table 2.

	a	**b**	**c**	**d**	**e**
a	0	2	1	–	–
b	1	0	2	–	–
c	2	1	0	2	1
d	–	–	1	0	2
e	–	–	2	1	0

Table 2: Hedonic payoffs in Network Structure $\mathcal{B}$

We now claim that for these given values there does not exist a stable bilateral outcome in $\mathcal{B}$. For example, consider the outcome $\pi' = \{\mathbf{ab}, \mathbf{cd}, \mathbf{ee}\}$, then both agents **d** and **e** would prefer to form interaction **de** rather than being engaged with **c** and being autarkic, respectively. Other bilateral interaction patterns can be shown to be unstable as well.

This example illustrates the inherent stability of hunter-gatherer societies. In particular, Theorem 3.5 shows that all potential value-generating bilateral interactions have to be based on the assignment of two roles. Any violation of this institutional framework—such as considered in Figure 2—would make it impossible to guarantee existence of a stable interaction pattern for any arbitrary hedonic utility profile. Thus, the institutional roles in a hunter-gatherer society provide a framework that promotes stability (Theorem 3.6).

This example also illustrates that the introduction of hierarchical leadership and evolving to a more advanced institutional framework can lead to inherent and persistent instability vis-à-vis this basic bilateral activity pattern. Next we see that such leadership, however, facilitates stability when more complex forms of interaction are established.

1.2 Multilateral interaction

We consider multilateral interactions that require the active involvement of a *middleman*, who brings together a group of economic agents that forms a local community. The middleman may be seen as intermediating, coordinating, or managing the economic interaction between at least two other agents.

It is important to note that which agent assumes the role of middleman is endogenous in our framework. Similarly to the bilateral case, discussed above, in this multilateral network economy, a middleman can only engage with other agents if there exist potential relationships between them. Furthermore, we assume that the economic values generated in these multilateral exchanges are again expressed as hedonic utilities.

We thus arrive at a network economy in which economic agents can engage into three forms of economic activities: Autarkic self-provision; bilateral interaction; and multilateral communal interaction through the facilitation of a middleman. Each of these three forms of interactions generates different hedonic utility levels to its participants.

The necessity of hierarchical leadership. We claim that in a hunter-gatherer economy, we can still guarantee stability through the introduction of a *chieftain.* Taking account of multilateral interactions, we again consider network structure $\mathcal{B}$ in Figure 2.

Following the definition sketched above, we introduce a multilateral interaction as *any* star-structured subnetwork of the depicted network in Figure 2. Thus, hunter **a** can act as a middleman for gatherer **b** and chieftain **c**, while chieftain **c** could principally be a middleman to all other agents. Thus, all middleman positions are independent of the assigned socio-economic roles of hunter, gatherer and chieftain.

To keep the discussion here more transparent, we assume that there are *no externalities*[6]: Every middleman's hedonic utility in a multilateral interaction is defined as the sum of the utilities generated in all bilateral interactions in which she engages, while other members only receive the hedonic payoff from interaction with the middleman. Thus, using the hedonic utilities from Table 2, if **b** is the middleman between **a** and **c**—represented as **bac**—she receives $u_b(\mathbf{bac}) = u_b(\mathbf{ab}) + u_b(\mathbf{bc}) = 3$. Agents **a** and **c** obtain the hedonic utility from the bilateral interaction with the middleman **b**: $u_a(\mathbf{bac}) = u_a(\mathbf{ab}) = 2$.

We devise a stability concept in which each agent participates in exactly one activity, being autarky; a bilateral household; or a multilateral communal interaction. In equilibrium, no agent has an incentive to join any other potentially formable activity. Such an equilibrium is called a *stable multilateral outcome.*

In network structure $\mathcal{B}$ with hedonic utility values represented in Table 2, representing a hunter-gatherer society with a chieftain, there now exists a stable multilateral outcome, namely the all-inclusive collaboration $\{\mathbf{cabde}\}$ centred on the chieftain **c** acting as

[6] Later in our formal analysis we study explicitly two forms of externalities: link-based and size-based.

the global middleman. Here, $u_c(\mathbf{cabde}) = 6$, $u_a(\mathbf{cabde}) = u_d(\mathbf{cabde}) = 1$ and $u_b(\mathbf{cabde}) = u_e(\mathbf{cabde}) = 2$. Now, hunter **a** would rather engage with gatherer **b**, but **b** would not agree due to the lowering of her payoff. Thus, the introduction of leadership in a hunter-gatherer society promotes multilateral stability. This is asserted in general in Theorem 5.7.

On the other hand, in network structure $\mathscr{A}$ representing a leaderless hunter-gatherer society, there is no such stable multilateral outcome. For the utility values in Table 1, take $\{\mathbf{ab}, \mathbf{ecd}\}$, then agents **a** and **b** obtain $u_a(\mathbf{ab}) = 2$ and $u_b(\mathbf{ab}) = 1$, respectively. On the other hand, agent **e** as a middleman receives $u_e(\mathbf{ecd}) = 2$, while (regular) members **c** and **d** receive $u_c(\mathbf{ecd}) = u_d(\mathbf{ecd}) = 2$. Now, agents **a** and **e** can mutually improve their positions and agent **b** will not suffer a loss from engagement in **abe**, where **a** acts as its middleman. Indeed, $u_a(\mathbf{abe}) = 3 > 2 = u_a(\mathbf{ab})$, $u_e(\mathbf{abe}) = 3 > 2 = u_e(\mathbf{ecd})$ and $u_b(\mathbf{abe}) = u_b(\mathbf{ab}) = 1$. Similarly, one can show that all other multilateral interaction patterns are unstable. This shows that an institutional rule such as the binary role pattern that guarantees stability in the context of bilateral activities no longer promotes stability when multilateral interactions are the form of established outcome.

It is clear from the discussion above that, as in the bilateral case, multilateral stability is directly emanating from institutional rules determining the nature of potential relationships among economic agents. Our main Equivalence Theorem 4.8 determines technical conditions for multilateral stability. Theorems 5.3, 5.5 and 5.7 show that certain institutional arrangements such as the imposition of social hierarchy and leadership result in network structures satisfying the technical conditions for the emergence of stability.

In its full development, we consider different forms of stability that implement certain features of multilateral interaction and collaboration. Formally, we distinguish "open" from "closed" multilateral platforms: In the latter a middleman fully controls the admittance of agents, while in the former this control is limited.

The remainder of this paper is organised as follows. Section 2 introduces our institutional approach to economic interaction. Sections 3 and 4 derive technical results related to the stability in bilateral and network economies, respectively, allowing certain types of externalities. In Section 5 we draw parallels between our technical results and sustainable institutional rules. In particular we show how leadership and social hierarchies function as institutional stabilisers. Proofs of the main theorems are collected in three appendices.

2 Networks and institutions

In this section we introduce some fundamental concepts from social network analysis[7] allowing us to develop key concepts in our institutional approach to describing networked

[7] For a comprehensive overview of concepts from network analysis and network formation theories, we refer to Jackson and Wolinsky (1996), Jackson (2003), Jackson (2008) and Newman (2010).

economic activities.

Our main postulate is that all economic interaction is principally relational and occurs in a network that is based on certain institutional principles. Now, an *economic activity* is abstractly defined as any interaction between linked agents that generates a hedonic utility value for each of its participants. We emphasise that from this perspective the economy solely consists of relational activities that are constrained by the imposed institutional rules guiding economic interaction. For example, in a hunter-gatherer economy only households as bilateral interactions between hunters and gatherers are considered, based on the social institution of a primitive social division of labour.

Throughout we work with a finite set of *economic agents* denoted by $N = \{1, \dots, n\}$ where $n \geqslant 3$. These economic agents can engage in three different relational economic activities that generate individual economic values to the participants.

Autarkic activities. The first and most primitive form of economic activity is that of *economic autarky* of an economic agent $i \in N$, denoted by ii, in which agent i engages in home production only. Thus, we arrive at the *class of all autarkic activities* $\Omega = \{ii \mid i \in N\}$. The hedonic utility level $u_i(ii)$ of an agent $i \in N$ in autarky ii is interpreted as the generated subsistence level for that agent.

Bilateral interactions. A second type of economic activity is that of a *bilateral interaction* in the sense that two agents i and j engage into some bilateral activity such as householding, commodity exchange or service provision for monetary compensation that generates hedonic utility values for both of these agents.

Formally, for any pair of agents $i, j \in N$ with $i \neq j$ the mathematical expression $ij = \{i, j\}$ represents a bilateral economic activity involving agents i and j.[8] Assuming institutional restrictions on bilateral interaction formation, the network structure Γ represents the institutionally feasible bilateral interactions between agents in N, where

$$\Gamma \subseteq \Gamma_N = \{ij \mid i, j \in N \text{ and } i \neq j\}. \tag{1}$$

Throughout we assume that for every agent $i \in N$ there is some $j \neq i$ with $ij \in \Gamma$.

In terms of our framework one can think of the pair (Ω, Γ) as the institutional matrix in which all economic activities emerge. Autarkic and bilateral activities form the *simple interaction structure* $\Delta^m = \Omega \cup \Gamma$. For any sub-structure $H \subseteq \Delta^m = \Omega \cup \Gamma$ we denote

$$N(H) = \{i \in N \mid \text{There is some } j \neq i \text{ such that } ij \in H\} \tag{2}$$

as the set of economic agents that are bilaterally engaged within the sub-structure H. It

[8] We remark here that $ij = ji$. Note that if $i = j$, the relational activity ii represents again the economic autarky of agent i.

is easy to see that $N(H) = N(H \setminus \Omega)$. Also, for every $H \subseteq \Gamma$, if $H \neq \varnothing$, then $N(H) \neq \varnothing$. Finally, due to the assumptions made, it holds that $N(\Delta^m) = N(\Gamma) = N$.

We define a *path* between any two distinct agents $i \in N$ and $j \in N$ in $H \subseteq \Gamma$ as a sequence of distinct agents $P_{ij}(H) = (i_1, i_2, \ldots, i_m)$ with $i_1 = i$, $i_m = j$, $i_k \in N$ and $i_k i_{k+1} \in H$ for all $k \in \{1, \ldots, m-1\}$. We define a *cycle* in H to be a path of an agent from herself to herself which contains at least two other distinct agents, *i.e.*, a cycle in H from i to herself is a path $C = (i_1, i_2, \ldots, i_m)$ with $m \geqslant 4$, $i_1 = i_m = i$, $i_k \in N$ distinct for all $k = 1, \ldots, m-1$, and $i_k i_{k+1} \in H$ for all $k \in \{1, \ldots, m-1\}$. The length of the cycle C is denoted by $\ell(C) = m - 1 \geqslant 3$. A sub-structure $H \subseteq \Gamma$ is called *acyclic* if H does not contain any cycles.

Agent i's *neighbourhood* in sub-structure H is defined as $N_i(H) = \{j \in N \mid ij \in H\}$. Note here that if $i \in N_i(H)$, then $ii \in H$. Also, by the definition of the bilateral interaction structure Γ, it holds that $N_i(\Gamma) \neq \varnothing$ for any $i \in N$. We can also express the neighbourhood of an agent within an arbitrary structure $H \subseteq \Delta^m$ in terms of its link based analogue, *i.e.*, $L_i(H) = \{ij \in H \mid j \in N_i(H)\} \subseteq H$. Therefore, $L_i(\Delta^m) = \{ii\} \cup L_i(\Gamma)$ is the set of feasible simple activities that i can potentially participate in.

Multilateral interactions. Extending the setting of a simple interaction structure (Ω, Γ) we introduce a third form of relational economic activity, that of a multilateral interaction. Such complex activities are assumed to be centred around a "middleman", representing an agent who acts as a hub in the network structure of this activity. In particular, a middleman brings together a number of economic agents with whom she already has an established bilateral relationship. This is formalised as follows:

Definition 2.1 *Let $\Gamma \subseteq \Gamma_N$ be a network structure on N.*
*A **multilateral interaction** in Γ is a sub-structure $G \subseteq \Gamma$ such that $|N(G)| \geqslant 3$ and there is a unique agent $i \in N(G)$ such that $N_i(G) = N(G) \setminus \{i\}$ and that for all other agents $j \in N(G) \setminus \{i\}$ it holds that $N_j(G) = \{i\}$. The agent i is called the **middleman** of the multilateral interaction G, denoted by $\mathcal{K}(G) \in N(G)$.*

Thus, a multilateral interaction has at least three members that form an explicit star structure in Γ. Hence, a multilateral interaction has a relational centre, the middleman, binding and coordinating all constituting bilateral relations in this activity. Throughout, we use the convention that the middleman is listed first and, so, a multilateral interaction G is described by $ij_1 \cdots j_m$, where $i = \mathcal{K}(G)$ and $N_i(G) = \{j_1, \ldots, j_m\}$.

A multilateral interaction might also be interpreted as a mathematical expression of a multi-sided interaction *platform* provided by its middleman in the sense of Hagiu and Wright (2011).[9] However, our abstract concepts also includes interpretations of multilateral

[9] Hagiu and Wright (2011, page 7) define a multi-sided platform as "an organization that creates value primarily by enabling direct interactions between two (or more) distinct types of affiliated customers".

interactions as clubs (Buchanan, 1965) or local authorities (Tiebout, 1956).

Using this definition, we introduce some auxiliary concepts and notation.

Definition 2.2 *Let $\Gamma \subseteq \Gamma_N$ be some network structure. The collection of all multilateral interactions in Γ is called the* ***multilateral extension*** *of Γ given by*

$$\Sigma(\Gamma) = \{G \mid G \subseteq \Gamma \textit{ is a multilateral interaction in } \Gamma\} \tag{3}$$

The triple $(\Omega, \Gamma, \Sigma(\Gamma))$ is referred to as a ***feasible activity structure*** *on N consisting of all autarkies $G_1 \in \Omega$, all feasible bilateral interactions $G_2 \in \Gamma$, and all multilateral interactions $G_3 \in \Sigma(\Gamma)$. Finally, we let $\Delta = \Omega \cup \Gamma \cup \Sigma(\Gamma)$.*

We emphasise that every multilateral interaction in $\Sigma(\Gamma)$ is institutionally feasible in Γ and, as such, the feasible activity structure $(\Omega, \Gamma, \Sigma(\Gamma))$ is a mathematical representation of all institutionally feasible interactions in the economy. This structure acts as a representation of the institutional rules that govern the interaction in the economy as a whole.

Finally, we remark that we can now introduce the *set of middlemen* in Γ as the collective of middlemen of all multilateral activities in $\Sigma(\Gamma)$:

$$\mathcal{K}(\Gamma) = \{i \in N \mid i = \mathcal{K}(G) \text{ for some } G \in \Sigma(\Gamma)\}. \tag{4}$$

It is clear that agents in $\mathcal{K}(\Gamma) \subset N$ play a crucial role in the formation of interaction structures in the economy. These agents represent therefore a class of potential *entrepreneurs* in the economy.

Institutions and activity structures. We investigate network interaction structures that emerge from the application of certain institutional rules. Here, an *institutional rule* is interpreted as a prescription that describes how a bilateral interaction structure Γ is constructed. Thus, institutions are interpreted as guides in the creation of interaction infrastructures. Formally, this can be expressed as follows.

Definition 2.3 *A network structure $\Gamma \subseteq \Gamma_N$* ***conforms to institutional rule*** *$\mathcal{I}$ if $\mathcal{I}$ prescribes exactly how the bilateral interactions $ij \in \Gamma$ between agents $i, j \in N$ are formed.*

Within a network structure Γ that conforms to an institutional rule $\mathcal{I}$ we can now apply the definition of a multilateral interaction to investigate any multi-agent collaboration that might emerge through the application of the institutional rule $\mathcal{I}$. In this regard the institutional rule $\mathcal{I}$ naturally restricts bilateral as well as multilateral interaction among agents in the economy.

3 Stability in bilateral network economies

In this section we discuss stability in an economy with autarkic and bilateral interactions only, extending the model of a matching economy of Gilles, Lazarova, and Ruys (2007).

Throughout we assume that every individual $i \in N$ has complete and transitive preferences over her set of feasible simple activities $L_i(\Delta^m) = \{ii\} \cup L_i(\Gamma) \subseteq \Delta^m = \Omega \cup \Gamma$ in which she can engage. We represent these preferences by a *hedonic utility function* $u_i^m \colon L_i(\Delta^m) \to \mathbb{R}$. Let $u^m = (u_1^m, \ldots, u_n^m)$ denote the resulting *hedonic utility profile.*

Definition 3.1 *A* ***bilateral economy*** *is defined as a triple* $\mathbb{E}^m = (N, \Delta^m, u^m)$ *in which* N *is a finite set of individuals,* $\Delta^m = \Omega \cup \Gamma$ *is a simple activity structure on* N*, and* $u_i^m \colon L_i(\Delta^m) \to \mathbb{R}$*,* $i \in N$*, is a hedonic utility profile on* Δ^m*.*

The main hypothesis in our model is that each individual $i \in N$ activates *exactly one* of her activities in $L_i(\Delta^m)$.

Definition 3.2 *An* ***outcome*** *in a bilateral economy* $\mathbb{E}^m = (N, \Delta^m, u^m)$ *is a map* $\pi \colon N \to \Delta^m$ *such that*

(i) $\pi(i) \in L_i(\Delta^m)$ *for all* $i \in N$ *and*

(ii) $\pi(i) = ij$ *implies that* $\pi(j) = ij$ *for all* $i, j \in N$ *with* $i \neq j$*.*

We refer to outcomes in a bilateral economy as *bilateral outcomes.* A bilateral outcome π can equivalently be represented by the induced sub-structure in Δ^m

$$\pi(N) = \{\pi(i) \mid i \in N\}. \tag{5}$$

The set of all bilateral outcomes π in $\mathbb{E}^m$ is denoted by Π^m. We remark that by the imposed hypotheses and definitions, $\Pi^m \neq \varnothing$. In particular, $\Omega \in \Pi^m$ and, according to the assumptions made on Γ, for any agent $i \in N$, there exist some $\pi \in \Pi^m$ with $\pi(i) = ij$ for every $ij \in \Gamma$.

With slight abuse of notation, we use $u_i^m(\pi)$ to denote the hedonic utility that agent $i \in N$ receives under outcome $\pi \in \Pi^m$, *i.e.*, $u_i^m(\pi) = u_i^m(\pi(i))$.

We apply the standard assumptions of individual rationality (**IR**) and a no-blocking condition from matching theory (Roth and Sotomayor, 1990)—also known as "pairwise stability" (**PS**) (Jackson and Wolinsky, 1996)—to define a notion of stability in bilateral economies.

Definition 3.3 *An outcome* $\pi \in \Pi^m$ *is* ***stable*** *in the bilateral exchange economy* $\mathbb{E}^m = (N, \Delta^m, u^m)$ *if all bilateral interactions in* π *satisfy the following properties:*

IR: $u_i^m(\pi) \geqslant u_i^m(ii)$ *for all* $i \in N$*, and;*

PS: *There is no blocking bilateral interaction with regard to π, in the sense that for all $i, j \in N$ with $i \neq j$, $ij \in \Gamma$ and $\pi(i) \neq ij$ it holds that*

$$u_i^m(ij) > u_i^m(\pi) \text{ implies that } u_j^m(ij) \leqslant u_j^m(\pi). \tag{6}$$

For an economy to have persistent access to gains from organisation, its social structure has to *universally* admit stable outcomes. Hence, regardless what productive abilities and consumption preferences the individual economic agents hold—both represented here by their (hedonic) utility functions—a stable outcome has to exist in the corresponding bilateral economy.

Definition 3.4 *A network structure Γ on N* ***supports universal bilateral stability*** *if for every hedonic utility profile u^m on $\Delta^m = \Omega \cup \Gamma$ there exists at least one stable bilateral outcome in the bilateral exchange economy $\mathbb{E}^m = (N, \Delta^m, u^m)$.*
An institutional rule $\mathcal{I}$ is ***bilaterally stable*** *if all of the network structures Γ conforming to $\mathcal{I}$ support universal bilateral stability.*

Clearly, a network structure that supports universal bilateral stability implies that the economy supports stability regardless of the exact wealth patterns generated. In this regard, such a network structure reflects institutional features which promote and enhance the emergence of stable patterns of economic activities.

The next result identifies the necessary and sufficient conditions on Γ for universal bilateral stability. Similar conditions have already been established in the literature on matching markets (Sotomayor, 1996; Chung, 2000; Jackson and Watts, 2008).

Equivalence Theorem 3.5 *A network structure Γ on N supports universal bilateral stability if and only if Γ is bipartite in the sense that there is a partitioning $\{N_1, N_2\}$ of N such that*

$$\Gamma \subseteq N_1 \otimes N_2 = \{ \, ij \mid i \in N_1 \text{ and } j \in N_2 \, \} \, . \tag{7}$$

For a proof of this result we refer to Appendix A of the paper.

Theorem 3.5 has a clear interpretation that, without proof, is expressed formally in the following assertion.

Theorem 3.6 *Let $\mathcal{I}_B$ be the institutional rule that there exist two socio-economic roles $\mathcal{R}_1$ and $\mathcal{R}_2$ such that any relation $ij \in \Gamma$ is feasible if and only if i has role $\mathcal{R}_1$ and j has role $\mathcal{R}_2$. Then $\mathcal{I}_B$ is bilaterally stable.*

In particular, Theorem 3.6 implies that an institutional social division of labour based on a dichotomy of hunters and gatherers guarantees universal stability.

4 Stability in multilateral network economies

Next we extend the scope of our analysis to include multilateral interactions. Let $\Delta = \Omega \cup \Gamma \cup \Sigma(\Gamma)$ be a feasible activity structure on the population N. For $i \in N$ we introduce the set of all feasible activities in which agent i can participate as

$$\mathcal{A}_i(\Delta) = \{ii\} \cup \{ij \mid ij \in \Gamma\} \cup \{G \mid G \in \Sigma(\Gamma) \text{ and } i \in N(G)\}. \tag{8}$$

Also, we let $\mathcal{A}(\Delta) = \cup_{i \in N} \mathcal{A}_i(\Delta)$.

For any economic agent $i \in N$, her preferences are again represented as a hedonic utility function $u_i \colon \mathcal{A}_i(\Delta) \to \mathbb{R}$. Now $u = (u_1, \ldots, u_n)$ is a profile of hedonic utility functions for all agents in N.

Definition 4.1 *A hedonic utility function $u_i \colon \mathcal{A}_i(\Delta) \to \mathbb{R}$ for agent $i \in N$ is **regular** if for all agents $i, j, k \in N$ such that $ij, ik \in \Gamma$ and $ijk \in \Sigma(\Gamma)$, $u_i(ij) \geqslant u_i(ii)$ and $u_i(ik) \geqslant u_i(ii)$ imply that $u_i(ijk) \geqslant u_i(ii)$.*
We denote by $\mathcal{U}$ the class of all regular hedonic utility profiles on Δ.

Regularity of the hedonic utility functions requires that if a multilateral interaction is composed of individually rational bilateral interactions only, then this multilateral interaction has to be individually rational as well. This seems a rather mild and natural hypothesis.

Now a *network economy* is defined as a structure Δ of feasible activities—autarky, bilateral, as well as multilateral—and a regular hedonic utility function profile:

Definition 4.2 *A **network economy** is a triple $\mathbb{E} = (N, \Delta, u)$ in which N is a finite set of economic agents, $\Delta = \Omega \cup \Gamma \cup \Sigma(\Gamma)$ is a feasible activity structure, and $u \in \mathcal{U}$ is a profile of regular hedonic utility functions $u_i \colon \mathcal{A}_i(\Delta) \to \mathbb{R}$, $i \in N$.*

As before, we assume that agents participate in exactly one activity.

Definition 4.3 *A **multilateral outcome** in $\mathbb{E} = (N, \Delta, u)$ is a map $\lambda \colon N \to \Delta$ such that $\lambda(i) \in \mathcal{A}_i(\Delta)$ and $\lambda(i) = G \in \Gamma \cup \Sigma(\Gamma)$ implies that $\lambda(j) = G$ for all $j \in N(G)$.*

A multilateral outcome λ now *generates* a corresponding partitioning of N given by $\Lambda = (G_1, \ldots, G_m) \equiv \lambda(N) \subseteq \Delta$.

A multilateral outcome is now stable if it satisfies certain standard stability conditions from matching theory (Roth and Sotomayor, 1990), network formation theory (Jackson and Wolinsky, 1996), and Tiebout equilibrium theory for club economies (Gilles and Scotchmer, 1997). Here, we introduce two notions of stability reflecting two forms of assumed control of middlemen over the membership of the provided multilateral interactions.

Definition 4.4 *Let $\mathbb{E} = (N, \Delta, u)$ be a network economy.*

(a) *A multilateral outcome $\lambda^\star \colon N \to \mathcal{A}(\Delta)$ generating $\Lambda^\star = (G_1^\star, \dots, G_m^\star)$ is **stable** in $\mathbb{E}$ if for every $p \in \{1, \dots, m\}$ the activity $G_p^\star \in \Lambda^\star$ satisfies the* individual rationality *IR and two* pairwise stability *conditions PS and PS* as specified below:*

IR *For all $i \in N(G_p^\star)$ it holds that $u_i(G_p^\star) \geqslant u_i(ii)$;*

PS *For all distinct agents $i \in N(G_p^\star)$ and $j \in N(G_q^\star)$ with $q \in \{1, \dots, m\}$ and $ij \in \Gamma$, $ij \notin G_p^\star \cap G_q^\star$:*

$$u_i(ij) > u_i(G_p^\star) \quad \textit{implies} \quad u_j(ij) \leqslant u_j(G_q^\star); \tag{9}$$

PS* *For all distinct agents $i \in N(G_p^\star)$ and $j \in N(G_q^\star)$ with $q \in \{1, \dots, m\}$ with $ij \in \Gamma$, $ij \notin G_p^\star \cap G_q^\star$ and either $j = \mathcal{K}(G_q^\star)$ or $G_q^\star \in \Gamma$:*

$$u_i(G_q^\star \cup \{ij\}) > u_i(G_p^\star) \quad \textit{implies} \quad u_j(G_q^\star \cup \{ij\}) \leqslant u_j(G_q^\star). \tag{10}$$

(b) *A multilateral outcome $\lambda^\star \colon N \to \mathcal{A}(\Delta)$ generating $\Lambda^\star = (G_1^\star, \dots, G_m^\star)$ is **strongly stable** in $\mathbb{E}$ if $\lambda^\star$ is stable—satisfying IR, PS and PS*—in $\mathbb{E}$ and, additionally, for every $p \in \{1, \dots, m\}$ the activity $G_p^\star \in \Lambda^\star$ satisfies* Reduction Proofness *[RP]:*

RP *If $G_p^\star$ is a multilateral interaction, i.e., $G_p^\star \in \Sigma(\Gamma) \cap \Lambda^\star$, it holds that for every sub-structure $G \subseteq G_p^\star$*

$$u_i(G) \leqslant u_i(G_p^\star) \tag{11}$$

where $i = \mathcal{K}(G_p^\star) = \mathcal{K}(G)$ is the middleman of both $G_p^\star$ and G.

Condition IR is a standard individual rationality condition that allows an individual to opt out of an economic activity if she is better off in autarky. Condition PS is similar to the condition for bilateral economies in Definition 3.3: It rules out blocking opportunities for pairs of agents who are not connected to each other in the present equilibrium.

Condition PS* rules out blocking opportunities for pairs of agents, at least one of whom can add a link without severing any of her existing links in the present equilibrium. Hence, such an agent is either a middleman or involved in a bilateral interaction. This condition requires that there are no two distinct agents who want to be linked to each other in a multilateral interaction in which one of them is a middleman.[10]

Both PS and PS* are concerned with the formation of new links by deviating agents. These conditions do *not* allow a middleman to block access to the multilateral interaction: Stability is founded on *openness* of such multilateral interactions in the sense that new

[10]Note that both middlemen and agents linked in a bilateral interaction, have multiple types of blocking opportunities: such agents can add a link with or without severing their current links. Such agents are subject to both (no blocking) conditions PS and PS*.

members can join the interaction if that is to their benefit. There are numerous economic activities and platforms that satisfy the principle of openness such as trading posts (stores), markets, open source communities, and many economic service provision cooperatives (clubs). In most of these cases, if entrants follow the house rules of the platform in question, they will not be excluded from participation.

The notion of strong stability is more demanding in the sense that Condition RP explicitly "closes" a multilateral interaction: The middleman can exclude any participant from the platform. In particular, RP states that agents are refused membership if that benefits the middleman. In practice, closed platforms include team production situations, exclusive clubs (guilds and unions), and particular supply chains in which an intermediary may discontinue a procurement relation with a primary input supplier.

Under (regular) stability, a middleman is merely a *coordinator* of the multilateral interaction that is fully open to participation. On the other hand, under strong stability a middleman has to be viewed as a *manager* of the activity under consideration, since she controls agents' access. We emphasise that strong stability implies stability, *i.e.*, management implies coordination, but that the reverse is not true.

Within the context of network economies we address the existence of stable multilateral outcomes for a class of hedonic utility profiles. Formally, we introduce:

Definition 4.5 *Let Γ be a network structure and let $\mathcal{U}^\star \subseteq \mathcal{U}$ be some given class of regular utility profiles on Γ.*

(a) *A network structure Γ **supports universal (strong) multilateral stability** on the class $\mathcal{U}^\star$ if for every utility profile $u \in \mathcal{U}^\star$ there exists a (strongly) stable multilateral outcome $\lambda^\star$ in the network economy $\mathbb{E} = (N, \Delta, u)$, where Δ is generated by Γ.*

(b) *An institutional rule $\mathcal{I}$ is **(strongly) stable** on the class $\mathcal{U}^\star$ if every network structure Γ conforming to $\mathcal{I}$ supports universal (strong) multilateral stability on $\mathcal{U}^\star$.*

Next we discuss analytical results for different classes of hedonic utility functions, particularly based on different forms of network externalities. The term "network externality" is used here to capture the dependence of one agent's utility on another agent's decision to link with a third one. Clearly, externalities can take many different forms. Therefore, any insights are based on certain assumptions on the nature of these externalities. Here we discuss two broad categories: link-based externalities and size-based externalities.

The most basic form of network externality is actually the complete absence of any external effect. This is formalised as follows.

Definition 4.6 *Let $\mathbb{E} = (N, \Delta, u)$ be a network economy and let $i \in N$. Then i's hedonic utility function $u_i \colon \mathcal{A}_i(\Delta) \to \mathbb{R}$ **exhibits no externalities** if for all multilateral interactions*

$G \in \mathcal{A}_i(\Delta) \cap \Sigma(\Gamma)$ *it holds that*

$$\begin{aligned} u_i(G) &= u_i(i\,\mathcal{K}(G)) && \text{if } i \neq \mathcal{K}(G) \\ u_i(G) &\geqslant \textstyle\sum_{j \in N(G)\setminus\{i\}} u_i(ij) && \text{if } i = \mathcal{K}(G) \end{aligned} \tag{12}$$

The class of utility profiles consisting of regular utility functions exhibiting no externalities is now denoted by $\mathcal{U}_N \subset \mathcal{U}$.

Stability properties in the case of no externalities are discussed below in combination with the case of link-based externalities.

4.1 Link-based externalities

A simple form of network externality is derived from the *connections model* developed in Jackson and Wolinsky (1996). Here we adapt this model to fit our setting. Suppose that two agents $i, j \in N$ are in a feasible value-generating relationship $ij \in \Gamma$. Now, in a network economy this value-generating interaction can be executed *either* directly as bilateral interaction ij *or* through the intermediation of a middleman $\mathcal{K}(G)$ in a multilateral interaction G with $i, j \in N(G)$. If the second form of interaction occurs, the generated benefits are discounted by a loss due to intermediation by middleman $\mathcal{K}(G)$.

Definition 4.7 *Let* $\mathbb{E} = (N, \Delta, u)$ *be a network economy and let* $i \in N$. *Then* i*'s hedonic utility function* $u_i \colon \mathcal{A}_i(\Delta) \to \mathbb{R}$ ***exhibits link-based externalities*** *if there is some discount factor* $0 < \delta < 1$ *with for all multilateral interactions* $G \in \mathcal{A}_i(\Delta) \cap \Sigma(\Gamma)$:

$$\begin{aligned} u_i(G) &= u_i(i\,\mathcal{K}(G)) + \textstyle\sum_{j \in N(G) \cap N_i(\Gamma)} \delta \cdot u_i(ij) && \text{if } i \neq \mathcal{K}(G) \\ u_i(G) &\geqslant \textstyle\sum_{j \in N(G)\setminus\{i\}} u_i(ij) && \text{if } i = \mathcal{K}(G) \end{aligned} \tag{13}$$

The class of utility profiles consisting of regular utility functions exhibiting link-based externalities is denoted by $\mathcal{U}_L \subset \mathcal{U}$.

Note that the case when the utility function exhibits no externalities is equivalent to the link-based externality formulation for discount factor $\delta = 0$.

We now address the existence of (strongly) stable multilateral outcomes for arbitrary utility profiles exhibiting no or link-based externalities. Our investigation of the necessary and sufficient conditions for the support of universal stability results into the identification of a relatively large class of network structures.

Equivalence Theorem 4.8

(i) *The network structure* Γ *supports universal multilateral stability on the class* $\mathcal{U}_N$ *of hedonic utility profiles exhibiting no externalities if and only if* Γ *satisfies the property that for every cycle* $C \subseteq \Gamma$: $\ell(C) = 3k$, *where* $k \in \mathbb{N}$ *is some integer.*

(ii) *The network structure* Γ *supports universal multilateral stability on the class* $\mathcal{U}_L$ *of hedonic utility profiles exhibiting link-based externalities if and only if* Γ *satisfies the property that for every cycle* $C \subseteq \Gamma \colon \ell(C) = 3k$, *where* $k \in \mathbb{N}$ *is some integer such that* $k \geqslant 2$.

(iii) *The network structure* Γ *supports universal strong multilateral stability on the class* $\mathcal{U}_L \cup \mathcal{U}_N$ *of hedonic utility profiles exhibiting no or link-based externalities if and only if* Γ *satisfies the property that for every cycle* $C \subseteq \Gamma \colon \ell(C) = 6s$, *where* $s \in \mathbb{N}$ *is some integer.*

A proof of Theorem 4.8 is relegated to Appendix B.

The assertions stated in Theorem 4.8 above imply that only in the absence of cycles of certain specified lengths in the network structure Γ there emerges universal stability. If there are link-based externalities, this includes triads. But if there are no externalities this condition is considerably weaker, allowing triads to be present in the network structure of the economy.

Theorem 4.8(iii) imposes a rather strong condition on the network structure in the economy. Importantly these conditions are the same for an economy without externalities as for economies with link-based externalities. Indeed, any cycle of length other than a 6-fold is excluded. In particular, there are no triads of length 3 present in the network, thus implying that the network only consists of weak ties in the sense of Granovetter (1973).

4.2 Size-based externalities

Next we consider a wider class of externalities based on the size of the multilateral interaction considered. Such externalities can be positive ("synergistic") or negative ("crowding"). In the literature on Tiebout and club economies such crowding externalities have already been investigated extensively (Conley and Wooders, 1997; Conley and Konishi, 2002).

For utility profiles with size-based externalities, the number of agents in a multilateral interaction determines the intensity of the externality. The identity of the middleman determines whether this externality is synergistic or crowding as well as its magnitude.

Definition 4.9 *Let* $\mathbb{E} = (N, \Delta, u)$ *be a network economy. The hedonic utility function* u_i *of* $i \in N$ ***exhibits size-based externalities*** *if for every multilateral interaction* $G \in \Sigma(\Gamma)$:

$$u_i(G) = \sum_{j \in N_i(G)} u_i(ij) + \alpha_c \cdot [\, \#N(G) - 2\,] \tag{14}$$

for all $i \in N(G)$, *where* $c = \mathcal{K}(G)$ *is* G*'s middleman and* $\alpha_c \in \mathbb{R}$ *is a middleman-specific synergy parameter.*

The class of regular hedonic utility profiles u *that exhibit side-based externalities is denoted by*

*$\mathcal{U}_S \subset \mathcal{U}$. The subclass of utility profiles that exhibit **synergistic externalities** with $\alpha_c > 0$ for all middlemen $c \in \mathcal{K}(\Gamma)$ is denoted by $\mathcal{U}_S^+ \subset \mathcal{U}_S$ and the subclass of utility profiles that exhibit **crowding externalities** with $\alpha_c < 0$ for all middlemen $c \in \mathcal{K}(\Gamma)$ is denoted by $\mathcal{U}_S^- \subset \mathcal{U}_S$.*

First we report that there exist network economies exhibiting size-based externalities in which there is no stable multilateral outcome. An example is presented below.

Example 4.10 Let $N = \{1,2,3,4\}$ and $\Gamma = \{12,23,34\}$ be the line network. Let $\alpha_2 = 200$ and $\alpha_3 = -50$. Let the hedonic utility profile be such that $u_1(12) = u_2(22) = u_3(33) = -100$, $u_1(11) = u_2(12) = 0$, $u_2(23) = u_4(34) = 100$, $u_4(44) = 90$, $u_3(23) = 60$, and $u_3(34) = 300$. Using the linear size-based externality formulation (14), we can compute the utility levels in the two possible multilateral interactions 213 and 324 in a straightforward manner: $u_1(213) = 100$, $u_2(213) = 300$, $u_3(213) = 260$, $u_2(324) = u_4(324) = 50$, and $u_3(324) = 310$.
We now claim that in this example there is no stable outcome. First, consider the outcome $(12,34)$. It is not stable since Condition PS* is not satisfied: $50 = u_2(324) > u_2(12) = 0$ and $310 = u_3(324) > u_3(34) = 300$. Also, since $-100 = u_2(22) < u_2(324) = 50$, Condition PS* is not satisfied for the outcome $(11,22,34)$. Next, $(11,324)$ is not stable since IR for agent 4 is not satisfied: $50 = u_4(324) < u_4(44) = 90$. Moving on, the outcome $(11,23,44)$ is not stable due to a violation of PS*: $0 = u_1(11) < u_1(213) = 100$ and $100 = u_2(23) < u_2(213) = 300$. Finally, $(213,44)$ is not stable due to a violation of PS: $260 = u_3(213) < u_3(34) = 300$ and $90 = u_4(44) < u_4(34) = 100$. Using the same reasoning, we find that $(12,33,44)$ and $(11,22,33,44)$ are not stable either. ♦

Second, stability may not be possible if we impose *crowding* externalities on all multilateral interactions. The following example illustrates this.

Example 4.11 Let $N = \{1,2,3\}$ and let $\Gamma = \{12,23\}$. Now assume $\alpha_2 = -2$. Let the utility profile be such that $u_i(ii) = 0$ for all $i = 1,2,3$ and $u_1(12) = u_2(12) = 3$, $u_2(23) = 4$, and $u_3(23) = 1$. Using the linear size-based externality formulation (14), we compute the utility levels for 213 in a straightforward manner: $u_1(213) = 1$, $u_3(213) = -1$, and $u_2(213) = 5$. We now claim that there is no stable multilateral outcome in this network economy.
To show this, first, consider $(12,33)$. This outcome is not stable due to a violation of PS: $3 = u_2(12) < u_2(23) = 4$ and $0 = u_3(33) < u_3(23) = 1$. Similarly, $(11,22,33)$ is not stable. Next, $(11,23)$ is not stable due to a violation of PS*: $0 = u_1(11) < u_1(213) = 1$ and $4 = u_2(23) < u_2(213) = 5$. Finally, (213) is not stable due to a violation of IR for agent 3: $-1 = u_3(213) < u_3(33) = 0$. ♦

Finally, we consider a 5-agent circular network structure. Here, the size-based externalities are assumed to be synergistic. However, the emergence of a Condorcet-like cycle in the economy prevents the emergence of the desired stability.

Example 4.12 Let $N = \{1,2,3,4,5\}$ and let $\Gamma = \{12,15,23,34,45\}$. Furthermore, let $\alpha_c = \alpha = 2$ for all potential middlemen $c \in \mathcal{K}(\Gamma) = N$. Let the utility levels for each simple activity be given by $u_i(ii) = 0$ for all $i \in N$, $u_1(12) = u_2(23) = u_3(34) = u_4(45) = 2$, $u_1(15) = u_2(12) = u_3(23) = u_4(34) = u_5(45) = 10$ and $u_5(15) = -1$. The utility levels in all possible multilateral interactions are computed in a straightforward manner from the linear size-based externality formulation: $u_5(125) = 1$, $u_1(213) = u_2(324) = u_3(435) = u_4(514) = 4$, $u_5(514) = 11$, $u_1(514) = u_2(125) = u_3(213) = u_4(324) = u_5(435) = 12$, and $u_1(125) = u_2(213) = u_3(324) = u_4(435) = 14$. One can easily check that also in this example there is no stable multilateral outcome. ♦

We conclude from these examples that the emergence of a stable multilateral outcomes is prevented if (1) there are crowding externalities for all or some multilateral interactions, or (2) there are cycles in Γ. However, if these conditions are ruled out, stability can be established.

Theorem 4.13 *If Γ is an acyclic network structure, then Γ supports universal multilateral stability on the class $\mathcal{U}_S^+$.*

A proof of this existence result can be found in Appendix C.

The impossibility of strong universal stability. Theorem 4.13 cannot be strengthened to the case of strong stability. The next example devises a simple case satisfying the conditions of Theorem 4.13 in which no strongly stable multilateral outcome can be constructed. Thus, in the presence of these externalities only economies with "open" multilateral economic activities can achieve stability.

Example 4.14 Let $N = \{1,2,3\}$ with $\Gamma = \{12,23\}$ and $\Sigma(\Gamma) = \{213\}$. We consider the hedonic utility profile with size-based externalities generated by $\alpha_2 = 2$ and $u_1(11) = u_3(33) = 0$, $u_2(22) = -4$, $u_1(12) = -1$, $u_2(23) = -3$, and $u_2(12) = u_3(23) = 1$. Hence, $u_1(213) = -1 + 2 = 1$, $u_2(213) = 1 - 3 + 2 = 0$, and $u_3(213) = 1 + 2 = 3$.

We now check that in this economy there is no strongly stable multilateral outcome: $\{11,23\}$ is not stable since agent 1 wants to join agent 2 in the multilateral interaction 213 and its middleman, agent 2, agrees; $\{12,33\}$ is not stable since IR is not satisfied for agent 1; $\{213\}$ is not strongly stable since its middleman, agent 2, prefers 12 over 213 and thus severs the participation of agent 3; and $\{11,22,33\}$ is not stable since agents 2 and 3 prefer the bilateral interaction 23 over being autarkic.

Although there is no strongly stable multilateral outcome in this network economy, the multilateral interaction $\{213\}$ forms a stable one. ♦

5 The stability of institutional arrangements

The discussion so far derives the analytical conditions for a network structure Γ to support various forms of universal stability. These conditions are stated in Equivalence Theorems 3.5, 4.8 and Theorem 4.13. Theorem 3.6 already clarified the case of bilateral institutional stability. Here we debate the institutions that impose stability through the network structures that conform to these institutional rules. In particular, we focus on three institutional arrangements or frameworks, namely that of a strict social hierarchy, a tiered leadership organisation and a (global) market place.

Strict social hierarchies. One particular class of social institutional arrangements that implies the satisfaction of the conditions of Equivalence Theorem 4.8 as well as Theorem 4.13 is that of a strict social hierarchy.

Definition 5.1 *A* ***strict social hierarchy*** *is an institutional rule $\mathcal{I}_{SH}$ such that there is some partitioning $\mathcal{T} = \{T_1, \ldots, T_K\}$ of N with $T_k \cap T_{k'} = \varnothing$ for all $k \neq k'$ and $\cup \mathcal{T} = N$ incorporating the following network structure rules:*

(i) *Let $i \in T_k$ for some $k \geqslant 2$, then (1) i has a feasible relationship ij with exactly one agent $j \in T_{k-1}$, (2) i can have any number of feasible relationships with agents in T_{k+1} and (3) i has no other relationships.*

(ii) *Let $i \in T_1$, then (1) i can have any number of feasible relationships with agents in T_2 and (2) i has no other relationships.*

The classes in $\mathcal{T}$ can be interpreted as *social classes*, where T_1 is the highest ("royal") class and T_K is the lowest ("serf") class. Every agent now has exactly one superior and any number of subordinates in the next class. There are no relationships that cross multiple tiers or classes.

A special case of a strict social hierarchy is a *global market place* in which there is one global market maker $m \in N$ who is interacting with all other agents $i \in N \setminus \{m\}$. Hence, a global market place is a strict social hierarchy with two classes $T_1 = \{m\}$ and $T_2 = N \setminus \{m\}$. One can interpret the market maker m as a global platform provider in which all other agents participate.

The following lemma provides the relationship between the strict social hierarchy $\mathcal{I}_{SH}$ rule and the network structures to which it corresponds. This lemma is stated as Theorem 2.3 in Brink and Gilles (1994). For a proof we refer to that paper.

Lemma 5.2 *A network structure Γ on N conforms to the strict social hierarchy $\mathcal{I}_{SH}$ if and only if Γ is acyclic.*

With reference to the equivalence theorems stated in Section 4, we can now immediately conclude the following assertion regarding strict social hierarchies.

Theorem 5.3 *The institution of the strict social hierarchy $\mathcal{I}_{SH}$ is strongly stable on $\mathcal{U}_N \cup \mathcal{U}_L$ and stable on $\mathcal{U}_S^+$.*

Piccione and Rubinstein (2007) show that a strict social hierarchy even guarantees outcomes that satisfy the first and second welfare theorems. This conforms with the wide-ranging stability properties identified in the assertion above.

Tertius gaudens hierarchies. Next we consider an institutional setting that weakens the strict social hierarchy and allows the formation of certain triads in the hierarchy. This is based on the application of Burt's *tertius gaudens* principle (Burt, 1992): In the presence of tension between two agents, a third agent can take control over the relational benefits and realise her most preferred outcome. This is the case when a manager exploits the competition of two subordinates for a promotion or when a broker benefits from the tension between a buyer and a seller by extracting all gains from trade.

The institutional rule reflecting this *tertius gaudens* principle can be introduced formally through a modification of $\mathcal{I}_{SH}$.

Definition 5.4 *A **Tertius Gaudens hierarchy** is an institutional rule $\mathcal{I}_{TH}$ such that there is some partitioning $\mathcal{T} = \{T_1, \ldots, T_K\}$ of N with $T_k \cap T_{k'} = \varnothing$ for all $k \neq k'$ and $\cup\mathcal{T} = N$ incorporating the following network formation rules:*

(i) *Let $i \in T_k$ with $k \geqslant 2$, then* (1) *i is linked with exactly one agent from the preceding hierarchical level T_{k-1} and* (2) *i has no links with agents from higher order levels T_{k-s} with $s \geqslant 2$;*

(ii) *Let $i \in T_k$, then i can have at most* one *link with an agent $j \in T_k$ from her own hierarchical level, and;*

(iii) *Let $i, j \in T_k$ with $k \geqslant 2$ be two linked agents, then there is some agent $h \in T_{k-1}$ who is linked to both i and j.*

An example of a network structure satisfying $\mathcal{I}_{TH}$ is shown on Figure 3 where agents are mapped into three hierarchical levels—dark grey, light grey, and white. Here, if two agents of the same hierarchical level are linked together, like, e.g., e and f at the lowest level, they also share the same "boss" located on the level above.

Here, a link across hierarchical levels represents an authority relation. By interpreting links between agents at the same hierarchical level to represent interaction based on substitutable skills among colleagues, we identify competitive tension among co-workers. Indeed, control and tension are the key notions underlying the principle of *tertius gaudens*: The control over lower ranked individuals is reinforced by divergent preferences between them. Thus, a higher ranked individual may induce more effort and better performance

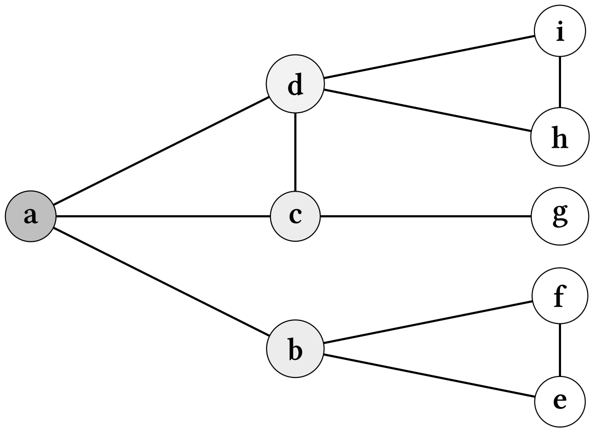

Figure 3: A Tertius Gaudens network structure

from her subordinates. In Figure 3 we identify two controlled or managed branches, b–e–f and c–d–g–h–i, that are based on these principles.

On the one hand, the institutional rule $\mathcal{I}_{TH}$ induces limited connectivity across branches. This, in turn, facilitates specialisation, provides opportunities for developing originality and innovation as any branch of the hierarchy can develop an independent mode of governance, and stimulates product building or information generation.

On the other hand, as stated, $\mathcal{I}_{TH}$ does not completely eliminate redundant links. In the presence of redundant links, the flow of information can still reach all agents in the organisation, even if some links fail. Redundant links may also provide higher speed of transmission of information along the organisation network and ensure sufficient level of compatibility across independently developing branches. In the example, one of the links among $\{ac, cd, ad\}$ is redundant, but functions as an insurance against the severance of the other links.

Equivalence Theorem 4.8 and the fact that Tertius Gaudens hierarchies result in networks in which all cycles are closed triads leads to the following assertion.

Theorem 5.5 *The Tertius Gaudens hierarchy $\mathcal{I}_{TH}$ is stable on $\mathfrak{U}_N$.*

Due to the nature of the networks resulting from $\mathcal{I}_{TH}$ this assertion cannot be extended or strengthened, in particular since Equivalence Theorem 4.8(ii) excludes triads from the network structure Γ.

Weak global market places. Finally we consider the introduction of market-makers into bilateral networks as discussed in the example of a hunter-gatherer economy as the emergence of a chieftain. A global market-maker is a unique individual economic agent who is linked directly to multiple other members in the population.

Definition 5.6 *A* ***weak global market place*** *is an institutional rule $\mathcal{I}_W$ such that there exists a mapping $r\colon N \to \{A, B, M\}$, where A, B and M are three distinct socio-economic* *roles—interpreted as two partner roles (A and B) and a market-maker role (M)—that satisfies the following rules:*

(i) *For all agents $i, j \in N$ with $i \neq j$ and $r(i) = r(j)$: $ij \notin \Gamma$;*

(ii) *There is at most one market maker, i.e., $\#\{i \in N \mid r(i) = M\} \leqslant 1$, and;*

(iii) *Each A-agent has at most one link with a B-agent and vice versa.*

Obviously a weak global market place is a global market place in which sparse local interactions are maintained. Thus, the global market maker M bridges local markets represented by bilateral interactions between trading partners of type A and B.

An example of a network conform $\mathcal{I}_W$ is shown in the right panel in Figure 4. This network structure is based on a construction method using a simple binary network structure depicted in the left hand panel to which a unique market maker g is added.

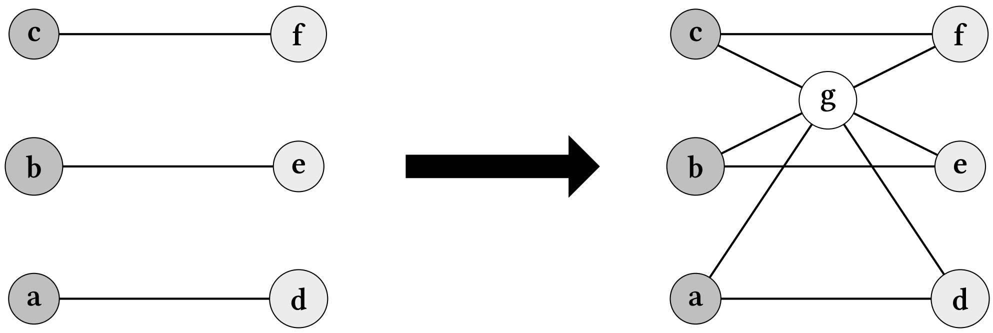

Figure 4: Construction of a weak global market place

In the left hand panel of Figure 4 we depict a network of bilateral exchanges between two types of agents: dark grey and light grey. This network is conform $\mathcal{I}_B$ and represents a situation of completely localised interaction. Clearly, this network exhibits structural holes in the sense of Burt (1992).

Next the market-maker g is introduced to create a global market place. The market-maker can be viewed as an entrepreneurial agent who exploits the presence of structural holes and invests in the links that will bridge these holes, thus, increasing opportunities for mutually beneficial trade.[11]

Applying Equivalence Theorem 4.8 and the fact that a weak global market place generates closed triads in the resulting network structures, we derive the following assertion.

[11]Note that here the market maker g may benefit from the tension in negotiations between any given pair of gray agents to extract rent by providing outside opportunities, using the *tertius gaudens* principle. In the right-hand panel, each market participant has a choice of engaging directly with her potential partner or execute her trade through the market maker. It is worth pointing out that in order to ensure the existence of stability the role of the market maker cannot be contested (Gilles and Diamantaras, 2013).

Theorem 5.7 *The weak global market place $\mathcal{I}_W$ is stable on $\mathcal{U}_N$.*

As before, this assertion cannot be strengthened.

Concluding remarks. What is clear from our analysis is that to render more complex economic outcomes stable for any distribution of preferences, the underlying network must satisfy a more complex set of properties. In the case of bilateral interactions, a necessary and sufficient characteristic is summarised in the rule of binary role assignment among economic agents. Whereas for stability with multilateral interactions, these conditions are sensitive with respect to the presence and type of externalities within a multilateral interaction and the discretionary power of the middleman to sever just one of her existing interactions and not any others. Moreover, adopting complex institutional rules – such as those shown appropriate for the functioning of more complex multilateral interactions – to govern more basic bilateral interactions may be ineffective and lead to instability.

Ruys (2015) extends the relational approach developed here to include relational capacities. An *enterprise* is then defined as an operator on a minimal structure of independent relational capacities. An enterprise's objective is to enhance its relational capacity. Multilateral interactions in this network structure specify the relational capacities that support its emergence and functioning as, for example, a "social" enterprise.

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Appendices

A Proof of Theorem 3.5

Here we show the necessary and sufficient conditions for the network structure to support universally bilateral stability. In Lemma 1, we establish a parallel with existing notions in the one-to-one matching literature.

Lemma 1 *Consider a bilateral economy $\mathbb{E}^m = (N, \Delta^m, u^m)$. Let the network structure Γ be bipartite in the sense that there exists a partitioning $\{N_1, N_2\}$ of N such that*

$$\Gamma \subseteq N_1 \otimes N_2 = \{\, ij \mid i \in N_1 \text{ and } j \in N_2 \,\}\,.$$

Then there exists a corresponding marriage problem in the sense of Gale and Shapley (1962) such that a stable matching in the marriage problem corresponds to a stable bilateral outcome in the bilateral economy $\mathbb{E}^m$.

Proof. A marriage problem as introduced by Gale and Shapley (1962) consists of two finite and disjoint sets of players M and W. Each agent $m \in M$ has complete and transitive preferences, $\succeq^M_m$, over $W \cup \{m\}$ and each agent $w \in W$ has complete and transitive preferences, $\succeq^W_w$, over $M \cup \{w\}$. A matching is a function $\mu : M \cup W \to M \cup W$ of order two, i.e., $\mu(\mu(i)) = i$, $\mu(m) \in W \cup \{m\}$ and $\mu(w) \in M \cup \{w\}$. A matching μ is stable if there is no (a) player $m \in M$ or $w \in W$ who prefers to be matched to herself than to her partner in μ, or (b) pair of distinct players (m, w) who are not matched by μ and $w \succeq^M_m \mu(m)$ and $m \succeq^W_w \mu(w)$. Notice that conditions (a) and (b) correspond to conditions IR and PS of Definition 3.3, respectively.
Consider a bilateral economy $\mathbb{E}^m = (N, \Delta^m, u^m)$ with a bipartite network structure Γ such that there exists a partitioning $\{N_1, N_2\}$ of N with

$$\Gamma \subseteq N_1 \otimes N_2 = \{\, ij \mid i \in N_1 \text{ and } j \in N_2 \,\}\,.$$

Let $\widetilde{\Gamma} = N_1 \otimes N_2 = \{\, ij \mid i \in N_1 \text{ and } j \in N_2 \,\}$. Next consider utility profile $\tilde{u}^m : \widetilde{\Gamma} \cup \Omega \to \mathbb{R}$ such that for all agents $i \in N$ and all matchings ij that satisfy the bipartite property but are *not* feasible, i.e., $ij \in \widetilde{\Gamma} \setminus \Gamma$, we set $\tilde{u}_i{}^m(ij) < u^m_i(ii)$, and for all matchings $ij \in \Delta^m$, we set $\tilde{u}^m = u^m$. Clearly, $\tilde{u}^m$ represents complete and transitive preferences on $\widetilde{\Gamma} \cup \Omega$.
Let $M = N_1$, $W = N_2$, and let preference profiles $\succeq^M$ and $\succeq^W$ be represented by hedonic utility functions $\phi^M_i : W \cup \{m\} \to \mathbb{R}$ with $\phi^M_i(N_i(ij)) = \tilde{u}_i(ij)$ for all $i \in M$ and all $ij \in \widetilde{\Gamma} \cup \Omega$ and $\phi^W_k(N_k(kl)) = \tilde{u}_k(kl)$ for all $k \in W$ and all $kl \in \widetilde{\Gamma} \cup \Omega$. The tuple $(M, W, \succeq^M, \succeq^W)$ defines a marriage problem.
Suppose μ^* is a stable matching in the marriage problem $(M, W, \succeq^M, \succeq^W)$. Consider, a bilateral outcome π^* in economy $\mathbb{E}$ such that $N_i(\pi^*(i)) = \mu^*(i)$ for all $i \in N$. Notice that $\pi^* \in \Delta^m$ follows from the stability of μ^*, which implies that for all $i \in M \cup W$, $\mu^*(i) \in N_i(\Delta^m)$, otherwise there is a contradiction to the stability of μ^* as there are two distinct players $k \in M$ and $l \in W$ with $\mu^*(k) = l$ and $kl \notin \Gamma$ such that k and l each prefer to be matched to themselves than to each other, i.e. $k \succeq^M_k l$ and $l \succeq^W_l k$ given by the construction of $\tilde{u}$, ϕ^M, and ϕ^W.
Lastly, we show that the stability of the matching function μ^* in the marriage problem implies the stability of the bilateral outcome π^* in the bilateral economy (N, Δ^m, u^m). The

proof follows by contradiction. Suppose the matching μ^* is stable and the bilateral outcome π^* is not stable. Therefore either IR or PS of Definition 3.3 must be violated.
Suppose, first, that IR does not hold and that there is an agent $i \in N$ such that $u_i(\pi^*) < u_i(ii)$. By construction, this implies that there is a player $i \in M$[12] such that $i \succeq_i^M \mu(i)$, which establishes a contradiction to the stability of μ^*.
Next, suppose that PS does not hold and that there are two distinct agents $i \in N_1$ and $j \in N_2$ with $ij \in \Gamma$ such that $u_i(ij) > u_i(\pi^*)$ and $u_j(ij) > u_j(\pi^*)$. By construction this implies that there are two distinct agents $i \in M$ and $j \in W$ with $\mu^*(i) \neq j$ such that $j \succeq_i^M \mu^*(i)$ and $i \succeq_j^W \mu^*(j)$ which contradicts to the stability of μ^*. ■

Proof of Theorem 3.5

If: Consider a bilateral economy $\mathbb{E}^m = (N, \Delta^m, u^m)$. Let the network structure Γ be bipartite in the sense that there exists a partitioning $\{N_1, N_2\}$ of N such that

$$\Gamma \subseteq N_1 \otimes N_2 = \{ \, ij \mid i \in N_1 \text{ and } j \in N_2 \, \} \, .$$

For any preference profile u^m, we can obtain a corresponding marriage problem as shown in Lemma 1. The existence of a stable matching in any marriage problem is shown by means of a constructive proof of Gale and Shapley (1962) and by means of a non-constructive proof in Sotomayor (1996). By analogy, this proves the existence of a stable bilateral outcome in bilateral economy $\mathbb{E}^m$ for any preference profiles u^m, given network structure Γ.

Only If: We show that if the network structure is not bipartite, there exists a preference profile for which there is no stable bilateral outcome in a bilateral economy.
Consider bilateral economy $\mathbb{E}^m = (N, \Delta^m, u^m)$ with $N = \{i, j, k\}$, and network structure $\Gamma = \{ij, ik, jk\}$. Consider the following preference profile: $u_i(ij) = u_j(jk) = u_k(ik) = 2$, $u_i(ik) = u_j(ij) = u_k(jk) = 1$, and $u_l(ll) = 0$ for all $l \in \{i, j, k\}$. It is easy to see that there is no stable bilateral outcome in this economy. For example, consider the outcome $\pi(i) = \pi(j) = ij$ and $\pi(k) = kk$. It is not stable because pairwise stability is not satisfied: $u_k(jk) > u_k(kk)$ and $u_j(jk) > u_j(ij)$. Similarly, one can show that no other bilateral outcome is stable.

This completes the proof of Theorem 3.5

B Proof of Theorem 4.8

The following Lemmas state two intermediate results. Throughout we let $\mathbb{E} = (N, \Delta, u)$ be some network economy. As before let $\Delta^m = \Omega \cup \Gamma$ be a simple interaction structure on N and let $u \in \mathcal{U}$ be an arbitrary profile of regular utility functions. Also, let

$$B_i(\Delta^m, u) = \{ \, j \in N \mid ij \in \Delta^m \text{ and } u_i(ij) \geqslant u_i(ik) \text{ for all } k \in N \text{ with } ik \in \Delta^m \, \} \tag{15}$$

be the *set of most preferred partners* of agent i for all $i \in N$.

Lemma 2 *Let the network structure Γ be acyclic. Then there is a pair of agents $i, j \in N$ with $i \neq j$ such that $j \in B_i(\Delta^m, u)$ and $i \in B_j(\Delta^m, u)$.*

[12] Here we assume, without loss of generality, that $i \in M$. If we were to assume, instead, that $i \in W$ the argument follows analogously.

Proof. If there is some agent $i \in N$ with $i \in B_i(\Delta^m, u)$ the assertion is obviously valid. Next assume that for every agent $i \in N$ it holds that $i \notin B_i(\Delta^m, u)$ and the second part of the assertion is not true. Then for all agents $i, j \in N$ with $i \neq j$ such that $j \in B_i(\Delta^m, u)$ it holds that $i \notin B_j(\Delta^m, u)$. Consider agent $i \in N$ and without loss of generality we may assume that the set of most preferred agents is a singleton, i.e., $B_i(\Delta^m, u) = \{j\}$. So, it must hold that $j \neq i$. Next, consider the set of most preferred partners of agent j. Without loss of generality we again may assume that B_j is a singleton, say $B_j(\Delta^m, u) = \{k\}$. It must again hold that $k \notin \{i, j\}$. Subsequently, consider the set of most preferred partners of agent k. Without loss of generality we again assume uniqueness, say $B_k(\Delta^m, u) = \{l\}$. It must be that $l \notin \{j, k\}$, moreover $l \neq i$ otherwise Γ contains a cycle. Hence, $l \notin \{i, j, k\}$. By continuing this process in a similar fashion, given that the player set N is finite, we construct a cycle. Therefore, we have established a contradiction. ■

Lemma 3 *Let (N, Δ, u) be a network economy and let Γ be an acyclic network structure. Then all paths between any two agents in N contain the same set of agents.*

Proof. The statement follows immediately from the fact that the network structure Γ contains no cycles. It is clear that if there were two distinct paths that connect two agents, these two paths would constitute a cycle. ■

Next we proof the assertions stated in Theorem 4.8. Notice that the presentation of the proof is in reversed order. This is because condition (iii) imposes to the most stringent requirements on the network structure whereas condition (i) imposes the least stringent.

Proof of Theorem 4.8(iii)

If: Consider a network economy $\mathbb{E} = (N, \Delta, u)$ such that $u \in \mathcal{U}_N \cup \mathcal{U}_L$. We consider two separate cases: (I) when Γ is acyclic; and (II) when Γ contains a cycle with an even number of connected agents that is a multiple of 3.
Let $S \subseteq N$ be some subset of economic agents. Then we denote by

$$\Gamma(S) = \Delta^m \cap \{ij \mid i, j \in S\}$$

the network structure and autarkic positions restricted to the subset S. In addition we use the operator $\oplus$ to denote an addition of an interaction to a given (partial) multilateral outcome, e.g. given the partial multilateral outcome $\Lambda = \{\{ijk\}, \{ll\}\}$, $\Lambda \oplus \{ih\} = \{\{ijhk\}, \{ll\}\}$. Finally, we slightly abuse notation and given a (partial) multilateral outcome Λ, we denote by $N(\Lambda)$ all agents that are part of this outcome, i.e. they are part of an autarky, or bilateral, or multilateral interaction in Λ. Using these auxiliary notations we proceed with the proof of the two cases.

Case I: Suppose Γ is acyclic. Thus, according to Lemma 3, no agent in a multilateral interaction gains any utility from having an indirect link via the middleman with a third agent. This is because for any two agents in a multilateral interaction, none of whom is a middleman, the only connecting path between them is via the middleman, and, therefore their direct link is not an element of the activity structure in this network economy. In this case the utility function in Equation (13) simplifies to:

$$u_i(G) = \sum_{j \in N_i(G)} u_i(ij), \tag{16}$$

for all $i \in N$ and all $G \in \mathcal{A}_i(\Delta)$.
We now devise an algorithm to construct a stable multilateral outcome in the economy $\mathbb{E}$ introduced above. This construction consists of several steps and collects agents in various multilateral interactions such that there are no possibilities for profitable deviations of all partners involved in the deviation.
We initiate the algorithm by setting N the set of agents, $\Gamma_1 = \Delta^m$ (the set of links that can be used in the construction of the outcome), $\Lambda_1 = \varnothing$ is a partial multilateral outcome and $K_1 = \varnothing$ is the set of agents who are active in outcome Λ and can act as middlemen. We now proceed by constructing the desired strongly stable multilateral outcome in a step-wise fashion:

Let N, $\Gamma_k \neq \emptyset$, Λ_k, K_k be given for k. We now proceed by constructing these elements for step $k+1$.

Take two agents $i \in N$ and $j \in N$ (notice that it is possible for $i = j$) such that $i \in B_j(\Gamma_k, u)$ and $j \in B_i(\Gamma_k, u)$. Such two agents exist for any non-empty $\Gamma_k \subseteq \Gamma$ by Lemma 2.

If $i = j$, then we define

$$\Lambda_{k+1} = \Lambda_k \cup \{ij\}; \tag{17}$$
$$\Gamma_{k+1} = \Gamma_k \setminus L_i(\Delta^m) \; ; \tag{18}$$
$$K_{k+1} = K_k. \tag{19}$$

Thus, in (17), we add the autarky $\{ii\}$ to the partial outcome Λ_k. In (18) we update the set of available interactions in Γ_k by eliminating all interactions that involve agent i. Last, we do not update the set of potential middlemen in the outcome Λ_{k+1} as the only new agent in this outcome, agent i, cannot add another link without exiting the autarkic state.
Subsequently we proceed to step $k+1$ in our construction process.

If $i \neq j$ and $i \notin K_k$ and $j \notin K_k$, then we define

$$\Lambda_{k+1} = \Lambda_k \cup \{ij\}; \tag{20}$$
$$\Gamma_{k+1} = \Gamma_k \setminus \Gamma\left(N(\Lambda_{k+1})\right) \; ; \tag{21}$$
$$K_{k+1} = K_k \cup \{i,j\}. \tag{22}$$

Thus, in (20), we add the bilateral interaction $\{ij\}$ to the partial outcome Λ_k. In (21) we update the set of interactions Γ_k by eliminating all links among agents who are already part of the outcome Λ_{k+1}, i.e. these are the autarkic relations of agents i and j, and all interactions of i and j with any other agent who is part of the outcome Λ_k. This is because by construction agents in Λ_k are connected to their most preferred partners, and thus, would not want to delete a link with their most preferred partner to join an interaction with i or j. Last, in (22) we update the set of potential middlemen in the outcome Λ_{k+1} by adding both agents i and j as they can add interactions to the existing one.
Subsequently we proceed to step $k+1$ in our construction process.

If $i \neq j$ and $i \notin K_k$ and $j \in K_k$, and $u_j(\Lambda_l \oplus \{ij\}) \leq u_j(\Lambda_k)$, then we define

$$\Lambda_{k+1} = \Lambda_k;$$
$$\Gamma_{k+1} = \Gamma_k \setminus \{ij\} \; ;$$
$$K_{k+1} = K_k.$$

This is the case when an agent wants to join a multilateral interaction but the middleman of this interaction is better-off if the interaction is not added. Thus, the only update is to eliminate the non-desirable interaction from the middleman's point of view from the set of possible interactions to be considered in the next step.
We proceed to step $k+1$ in our construction process.

If $i \neq j$ and $i \notin K_k$ and $j \in K_k$, and $u_j(\Lambda_l \oplus \{ij\}) > u_j(\Lambda_k)$, then we define

$$\Lambda_{k+1} = \Lambda_k \oplus \{ij\}; \tag{23}$$
$$\Gamma_{k+1} = \Gamma_k \setminus \{L_i(\Delta^m) \text{ for all } i \in N_j(\Lambda_{k+1})\}\ ; \tag{24}$$
$$K_{k+1} = K_k \setminus N_j(\Lambda_k). \tag{25}$$

This is the case when an agent wants to join a multilateral interaction and the middleman of this interaction is better-off when the interaction is added. Thus in (23) we add the interaction $\{ij\}$ to the existing multilateral or bilateral interaction in which agent j is involved in the partial outcome Λ_k. In (24) we remove from future consideration all interactions of all agents with whom j is connected because those agents cannot add any new interaction without deleting the one with j. For the same reason, we update the set of possible middlemen in (25) by removing all agents with whom j is connected in Λ_k. This is only important if j is involved in a bilateral interaction in the outcome Λ_k.

We proceed through the procedure until for some $k = \bar{k}$ we arrive at the situation that $\Gamma_{\bar{k}} = \varnothing$. (Note that such a $\bar{k} \leqslant |\Gamma|$ always exists.) Now consider $\Lambda^\star = \Lambda_{\bar{k}}$. First, since the procedure devised above assigns every agent to either an autarkic activity, a bilateral interaction, or a multilateral interaction, $\Lambda^\star$ is a multilateral outcome. Furthermore, each constructed interaction in $\Lambda^\star$ is based on either the optimality of an autarkic interaction, the optimality of a bilateral interaction, or the optimality of adding an interaction for a middleman. In the latter case, the form of the hedonic profiles given in (16) imply that the utilities generated in the constructed multilateral interactions in $\Lambda^\star$ are maximal under the imposed restrictions as well. Finally, this also guarantees that the middleman of multilateral interaction $G \in \Sigma(\Gamma) \cap \Lambda^\star$ does not have any incentives to break any relationships with members $i \in N(G)$. This implies, therefore, that the constructed multilateral outcome $\Lambda^\star$ is indeed strongly stable as required.

This concludes the Proof of Case I.

Case II: Suppose Γ contains a cycle $C = (i_1, \dots, i_m)$ of length $m-1 = 6s$ for some $s \in \mathbb{N}$. Since the length of the cycle is at least 6, it holds that for all distinct agents $i, j \in N$ who are connected in a multilateral interaction G via a middleman c ($ic, jc \in G$), $ij \notin \Gamma$, and, therefore, for all agents $i \in N$ the utility function $u_i \colon \mathcal{A}_i(\Delta) \to \mathbb{R}$ takes the form of equation (16).

Depending on the utility profile, we distinguish two sub-cases. In the first case the utility profile is such that the property of Lemma 2 is satisfied, and, thus, in all sub-sets of the simple interaction structure there exists a pair of agents who are in each other's set of most preferred partners. In the second case, the opposite is true, i.e., the utility profile is such that in at least one sub-set of the simple interaction structure all agents' most preferred partners do *not* have them in their respective sets of most preferred partners. Whereas in the former case we can follow the algorithm described in Case I to find a strongly stable outcome, in the latter case we show how the algorithm has to be augmented to find a strongly stable outcome.

Case II.a: Consider a utility function $u_i \in \mathcal{U}_L$ which exhibits link-based externalities. For the utility profile to satisfy the property of Lemma 2, it must be that (a) there exists an agent i_k with $k = 1, \ldots, m-1$ such that $i_k \in B_{i_k}(\Delta^m, u)$; or (b) there are two consecutive agents along the cycle $i_{k-1}, i_k \in C$ for some $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$ such that $i_{k-1} \in B_{i_k}(\Delta^m, u)$ and $i_k \in B_{i_{k-1}}(\Delta^m, u)$; or (c) there is a pair of agents one of whom is on the cycle and the other not, *i.e.*, $i_k \in C$ for some $k = 2, \ldots, m-1$ and $j \notin C$ such that $j \in B_{i_k}(\Delta^m, u)$ and $i_k \in B_j(\Delta^m, u)$. Then, we can use the algorithm described in Case I to construct a strongly stable assignment. The utility profile ensures that in each step k of the algorithm such that $\Gamma_k \neq 0$, there is (a) an agent $i \in N$ such that $i \in B_i(\Gamma_k, u)$; or (b) there is a pair of distinct agents $i, j \in N$ such that $j \in B_i(\Gamma_k, u)$ and $i \in B_j(\Gamma_k, u)$. Therefore, in this case, the presence of a cycle is immaterial for the implementation of the algorithm.
A similar argument applies to $u_i \in \mathcal{U}_N$.

Case II.b: Last, consider a profile of utility functions $u_i \in \mathcal{U}_L$ which satisfies the link-based externality property. For the utility profile to violate the property of Lemma 2, it must be that the utility profile is such that (a) for all agents i_k along the cycle $C = (i_1, \ldots, i_m)$ in Γ with $k = 1, \ldots, m-1$, $i_k \notin B_{i_k}(\Delta^m, u)$; (b) for all consecutive agents along the cycle $i_{k-1}, i_k \in C$ for some $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$, if $i_{k-1} \in B_{i_k}(\Delta^m, u)$, then $i_k \notin B_{i_{k-1}}(\Delta^m, u)$; and (c) for all pairs of distinct agents $i_k j \in \Delta^m$ of whom $i_k \in C$ is on the cycle and $j \notin C$ is not, it does *not* hold that $j \in B_{i_k}(\Delta^m, u)$ and $i_k \in B_j(\Delta^m, u)$.

For a utility function (16) to satisfy (a)-(c) above there are only three possibilities:[13]

(i) $u_{i_k}(i_k i_k) \leqslant u_{i_k}(i_{k-1} i_k) < u_{i_k}(i_k, i_{k+1}) \leqslant u_{i_k}(i_k i_{k-1} i_{k+1})$;

(ii) $u_{i_k}(i_k i_k) \leqslant u_{i_k}(i_{k-1} i_k) \leqslant u_{i_k}(i_k i_{k-1} i_{k+1}) < u_{i_k}(i_k, i_{k+1})$; or

(iii) $u_{i_k}(i_{k-1} i_k) < u_{i_k}(i_k i_k) < u_{i_k}(i_k, i_{k+1})$ for all $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$.

Suppose, the profile of utility function is as in (i), i.e., $u_{i_k}(i_k i_k) \leqslant u_{i_k}(i_{k-1} i_k) < u_{i_k}(i_k, i_{k+1}) \leqslant u_{i_k}(i_k i_{k-1} i_{k+1})$ for all $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$ and consider the partial multilateral outcome for the agents along the cycle $C = (i_1, \ldots, i_m)$ of length $m-1 = 6s$ for some $s \in \mathbb{N}$.

$$\Lambda_1 = \{\, \{i_2 i_1 i_3\},\ \{i_5 i_4 i_6\}, \ldots, \{i_{m-2} i_{m-3} i_{m-1}\} \,\} .$$

Here the agents along the cycles are organized in exactly $2 \times s$ multilateral interactions. Moreover, these agents have no blocking opportunities and all IR, PS, PS*, and RP conditions are satisfied. For example, consider agent i_3. This agent prefers to be in a bilateral interaction with agent i_4 than in a multilateral interaction with i_2 and i_1. Agent i_4, however, prefers to stay in the multilateral interaction where i_5 acts as a middleman than move to a bilateral interaction with i_3. Therefore, a multilateral outcome where the agents along the cycle are organized in $2 \times s$ multilateral interactions as defined in Λ_1 and all other agents outside the cycle are organized via the algorithm described in Case 1 is strongly stable.

Next, suppose the profile of utility function is as in (ii), i.e., $u_{i_k}(i_k i_k) \leqslant u_{i_k}(i_{k-1} i_k) \leqslant u_{i_k}(i_k i_{k-1} i_{k+1}) < u_{i_k}(i_k, i_{k+1})$ for all $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$.[14] In addition, consider the partial multilateral outcome, Λ_2, for all agents along the cycle

$$\Lambda_2 = \{\, \{i_1 i_2\},\ \{i_3 i_4\}, \ldots, \{i_{m-1} i_{m-2}\} \,\} .$$

[13] Recall that by assumption we have $u_{i_k}(i_k i_k) \leq u_{i_k}(i_{k-1} i_k)$ and $u_{i_k}(i_k i_k) \leq u_{i_k}(i_k, i_{k+1})$ implies that $u_{i_k}(i_k i_k) \leq u_{i_k}(i_k i_{k-1} i_{k+1})$. This assumption is innocuous in this context.

[14] Notice that for a utility function (16) to satisfy this relation, it must be that $u_{i_k}(i_{k-1} i_k) < 0$.

Λ_2 consists of exactly $3 \times s$ bilateral matchings. Notice that there are no blocking possibilities and all relevant no-blocking conditions IR, PS, and PS* are satisfied for all agents along the cycle. For example, consider agent i_2: i_2 prefers to be linked to i_3 either in bilateral interaction $i_2 i_3$ or in the multilateral interaction $i_3 i_2 i_4$ than to be matched to i_1. Agent i_3, however, prefers to be in a matching with i_4 than to add the link with i_2 and act as middleman. Therefore, a multilateral outcome, where the agents along the cycle are organized in $2 \times s$ multilateral interactions as defined in Λ_2 and all other agents outside the cycle are organized via the algorithm described in Case 1 is strongly stable.

Last, let the profile of utility function be as in (iii), i.e., $u_{i_k}(i_{k-1} i_k) < u_{i_k}(i_k i_k) < u_{i_k}(i_k, i_{k+1})$ for all $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$. In addition, consider partial outcome, Λ_3, where all, $m-1$, agents along the cycle are autarkic.

$$\Lambda_3 = \{ \{i_1 i_1\}, \{i_2 i_2\}, \ldots, \{i_{m-1} i_{m-1}\} \} .$$

Notice that again there are no blocking opportunities for any agent along the cycle. For example, consider agent i_1. Agent i_1 prefers to be in a bilateral interaction with agent i_2. Agent i_2, however, prefers to be in autarky than to be matched to i_1. Therefore, a multilateral outcome, where the agents along the cycle are organized in $m-1$ autarkies as defined in Λ_3 and all other agents outside the cycle are organized via the algorithm described in Case 1 is strongly stable.

Again a similar argument applies to $u_i \in \mathcal{U}_N$.

This completes the proof of Case II.

Only if: Let there be a strongly stable multilateral outcome in the network economy (N, Δ, u) for all $u \in \mathcal{U}_N \cup \mathcal{U}_L$. We show by contradiction the necessity of the condition that Γ contains no cycles, or that if it contains a cycle, it is a cycle with an even number of connected agents which is also a multiple of 3. We discuss two cases: the first case is when the length of the cycle is even but not a multiple of three, and the second one is when the length is odd. In both cases we identify utility profiles for which no strongly stable outcomes exist in the network economy.

Case I: Suppose that the network structure Γ contains a cycle $C = (i_1, i_2, \ldots, i_m)$ with $\{i_k i_{k+1}\} \in \Gamma$ for all $k = 1, \ldots, m-1$ and $m \geqslant 4$ and $m-1$ is an even number which is not a multiple of 3.

Now, consider a utility profile $u \in \mathcal{U}_L$ such that $u_j(i_k j) < u_j(jj)$ and

$$u_{i_k}(i_k j) < u_{i_k}(i_k i_k) < u_{i_k}(i_{k-1} i_k) < u_{i_k}(i_k i_{k+1}) < u_{i_k}(i_k i_{k-1} i_{k+1})$$

for all $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$ and all $j \in N_{i_k}(\Gamma) \setminus \{i_{k-1} i_{k+1}\}$. Let $\Lambda^\star$ be a strongly stable multilateral outcome in this network economy. Note that in the strongly stable outcome $\Lambda^\star$ the largest number of agents located along the cycle who can form a multilateral interaction that satisfies IR is three and that all of the agents in such a multilateral interaction are located along the cycle. In addition, since the length of the cycle is not a multiple of three, it must be that in $\Lambda^\star$ at least one agent is autarkic or at least two agents are in a bilateral interaction. We consider two sub-cases.

Case I.A: First, suppose that $\{i_k i_k\} \in \Lambda^\star$ for some $k = 1, \ldots, m-1$. Since $\Lambda^\star$ is a strongly stable outcome, the individual rationality condition is satisfied for all agents in N. Hence, agent i_{k-1} is in a state of autarky; or connected to agent i_{k-2} either in the bilateral interaction $g' = \{i_{k-1}i_{k-2}\}$, or the multilateral interaction $g'' = \{i_{k-2}i_{k-1}i_{k-3}\}$ with $i_0 = i_{m-1}$, $i_{-1} = i_{m-2}$, and $i_{-2} = i_{m-3}$. Moreover, using equation (16) to derive the utility of player i_{k-1} we note that $u_{i_{k-1}}(g') = u_{i_{k-1}}(g'')$. In all three cases the PS condition for agents i_{k-1} and i_k is violated: $u_{i_k}(i_{k-1}i_k) > u_{i_k}(i_k i_k)$ and $u_{i_{k-1}}(i_{k-1}i_k) > u_{i_{k-1}}(g'') = u_{i_{k-1}}(g') > u_{i_{k-1}}(i_{k-1}i_{k-1})$. Therefore, the strong stability of $\Lambda^\star$ implies that $\{i_k i_k\} \notin \Lambda^\star$ for any $i_k \in C$.

Case I.B: Next, suppose that strongly stable multilateral outcome $\Lambda^\star$ contains a bilateral interaction $\{i_k i_{k+1}\}$. Then, agent i_{k-1} is connected to agent i_{k-2} either through the bilateral interaction $g' = \{i_{k-2}i_{k-1}\}$, or the multilateral interaction $g'' = \{i_{k-2}i_{k-3}i_{k-1}\}$ with $i_0 = i_{m-1}$, $i_{-1} = i_{m-2}$, $i_{-2} = i_{m-3}$, and $i_{-3} = i_{m-4}$.[15] In all cases the no blocking condition PS* for agents i_{k-1} and i_k is violated: $u_{i_{k-1}}(i_k i_{k-1} i_{k+1}) = u_{i_{k-1}}(i_{k-1}i_k)$ as the bilateral interaction $i_{k-1}i_{k+1} \notin \Gamma$, and therefore equation (16) holds; furthermore by the definition of the utility function for Case I, we have $u_{i_{k-1}}(i_{k-1}i_k) > u_{i_{k-2}}(g') = u_{i_{k-2}}(g'')$ and $u_{i_k}(i_k i_{k-1} i_{k+1}) > u_{i_k}(i_k i_{k+1})$ with $k_{-1} = m-2$.
Hence, when Γ contains a cycle with an even number of connected agents which is not a multiple of three, there are such utility profiles that satisfy the link-based externality properties, for which there is no strongly stable outcome in the network economy.

Case II: Now suppose that the network structure Γ contains a cycle $C = (i_1, i_2, \ldots, i_m)$ with $i_k, i_{k+1} \in \Gamma$ for all $k = 1, \ldots, m-1$ and $m \geqslant 4$ and $m-1$ is an odd integer.
Now, consider a utility profile $u \in \mathcal{U}_N \cup \mathcal{U}_L$ such that $u_j(i_k j) < u_j(jj)$ and

$$u_{i_k}(i_k j) < u_{i_k}(i_k i_k) < u_{i_k}(i_{k-1}i_k) < u_{i_k}(i_k i_{k-1} i_{k+1}) < u_{i_k}(i_k i_{k+1})$$

for all $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$ and all $j \in N_{i_k}(\Gamma) \setminus \{i_{k-1}i_{k+1}\}$. Let $\Lambda^\star$ be a strongly stable multilateral outcome in this network economy. Note that in the strongly stable outcome $\Lambda^\star$ the largest number of agents located along the cycle that can form a multilateral interaction that satisfies the IR condition is three. In addition, since the length of the cycle is odd, in the outcome $\Lambda^\star$ there must be at least one agent who is autarkic or at least three agents who are in a multilateral interaction. We consider two sub-cases.

Case II.a: First, suppose that $i_k i_k \in \Lambda^\star$ for some $k = 1, \ldots, m-1$. Similar to Case I.A, we can show that the PS condition must be violated for agents i_{k-1} and i_k as $u_{i_k}(i_{k-1}i_k) > u_{i_k}(i_k i_k)$ and $u_{i_{k-1}}(i_{k-1}i_k) > u_{i_{k-1}}(\Lambda^\star) \geq u_{i_{k-1}}(i_{k-1}i_{k-1})$. Since $\Lambda^\star$ is strongly stable, then it cannot be that $\{i_k i_k\} \in \Lambda^\star$ for some $i_k \in C$.

Case II.b: Lastly, suppose that the multilateral interaction $\{i_k i_{k-1} i_{k+1}\} \in \Lambda^\star$ for some $k = 1, \ldots, m-1$ with $k_0 = i_{m-1}$ and $k_{m+1} = i_1$. In this case the RP condition is violated for agent i_k as $u_{i_k}(i_k i_{k-1} i_{k+1}) < u_{i_k}(i_k i_{k+1})$. Since $\Lambda^\star$ is strongly stable, then it cannot be that $\{i_k i_{k-1} i_{k+1}\} \in \Lambda^\star$ for some $i_{k-1}, i_k, i_{k+1} \in C$.
Hence, when Γ contains a cycle with an odd number of connected agents, there are such utility profiles that satisfy the link-based externality property, for which there is no strongly stable multilateral outcome in the network economy.
This completes the proof of Theorem 4.8(iii).

Proof of Theorem 4.8(ii)

If: Consider a network economy $\mathbb{E} = (N, \Delta, u)$ such that $u \in \mathcal{U}_L$ exhibits link-based externalities and $\delta \in (0,1)$. We consider two cases: (I) when Γ does not contain any cycle;

[15] Recall that Case I.a rules out that $\{i_{k-1}, i_{k-1}\} \in \Lambda^\star$.

(II) when Γ contains a cycle with a number of connected agents that is a multiple of 3 and greater than 3.

Case I: Suppose that Γ is acyclic. Since strong stability implies stability, the proof of Case I follows the steps in Case I of the proof of Theorem 4.8(iii).

Case II: Suppose that Γ has a cycle $C = (i_1, \ldots, i_m)$ with $m \geqslant 5$ and $m-1 = 3s$ for some $s \in \mathbb{N}$. Since the length of the cycle is at least 6, it holds, as in the proof of Theorem 4.8(iii), that for all distinct agents $i, j \in N$ who are connected in a multilateral interaction G via a middleman c ($ic, jc \in G$), $ij \notin \Gamma$, and, therefore, for all agents $i \in N$ the utility function $u_i \colon \mathcal{A}_i(\Delta) \to \mathbb{R}$ takes the form of equation (16).

Case II.a: First, consider a utility function $u_i \in \mathcal{U}_L$ which satisfies the link-based externality property, such that either (a) there exists an agent i_k with $k = 1, \ldots, m-1$ such that $i_k \in B_{i_k}(\Delta^m, u)$; or (b) there are two consecutive agents along the cycle $i_{k-1}i_k \in C$ for some $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$ such that $i_{k-1} \in B_{i_k}(\Delta^m, u)$ and $i_k \in B_{i_{k-1}}(\Delta^m, u)$; or (c) there is a pair of agents one of whom is on the cycle and the other not, *i.e.*, $i_k \in C$ for some $k = 2, \ldots, m-1$ and $j \notin C$ such that $j \in B_{i_k}(\Delta^m, u)$ and $i_k \in B_j(\Delta^m, u)$. Then, we can use the algorithm described in Case I to construct a stable multilateral outcome since the utility profile ensures that in any of the three cases described above, we can identify agents that fit the requirements stated in Lemma 2.

Case II.b: Next, consider a profile of utility functions $u_i \in \mathcal{U}_L$ which exhibits link-based externalities such that there is no agent i_k with $k = 1, \ldots, m-1$ such that $i_k \in B_{i_k}(\Delta^m, u)$, or there are no consecutive agents along the cycle $i_{k-1}, i_k \in C$ for some $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$ such that $i_{k-1} \in B_{i_k}(\Delta^m, u)$ and $i_k \in B_{i_{k-1}}(\Delta^m, u)$, nor is there a pair of agents one of whom is on the cycle and the other not, *i.e.*, $i_k \in C$ for some $k = 1, \ldots, m-1$ and $j \notin C$ such that $i_j \in B_{i_k}(\Delta^m, u)$ and $i_k \in B_j(\Delta^m, u)$.

Then, without loss of generality[16], we may assume that $\max\{u_{i_k}(i_k i_k), u_{i_k}(i_{k-1}i_k)\} < u_{i_k}(i_k i_{k+1})$, for all $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$.
Suppose, first, that the profile of utility functions is

$$u_{i_k}(i_k i_k) \leqslant u_{i_k}(i_{k-1}i_k) < u_{i_k}(i_k i_{k+1})$$

for all $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$. Then, a (partial) multilateral outcome Λ_1 can be introduced that consists of exactly s multilateral interactions of the type

$$\Lambda_1 = \{\, \{i_2 i_1 i_3\},\, \{i_5 i_4 i_6\}, \ldots, \{i_{m-2} i_{m-3} i_{m-1}\} \,\} \,.$$

Clearly, all no-blocking conditions IR and PS are satisfied for the agents in Λ_1. Consider, for example, agent i_3: i_3 prefers to in a bilateral interaction with agent i_4; i_4, however, has higher utility from the multilateral interaction $i_5 i_4 i_6$ than the bilateral $i_3 i_4$. Moreover the no-blocking condition PS* is automatically satisfied as no agent can add a link without deleting all existing links. Therefore, a multilateral outcome where all agents on the cycle are linked in multilateral interactions of the type described in Λ_1 and all agents not on the cycle are linked following the algorithm presented in Case I is stable.

[16]The statement is justified because by assumption we have ruled out the degenerate case $u_{i_k}(i_k i_k) \leq u_{i_k}(i_{k-1}i_k)$ and $u_{i_k}(i_k i_k) \leq u_{i_k}(i_k, i_{k+1})$ implies that $u_{i_k}(i_k i_k) \leq u_{i_k}(i_k i_{k-1} i_{k+1})$. In this context when all utility levels are negative and the number of connected agents on the cycle is odd, relaxing this assumption may lead to instability.

Last, suppose that the profile of utility function is given by

$$u_{i_k}(i_{k-1}i_k) \leq u_{i_k}(i_k i_k) < u_{i_k}(i_k i_{k+1})$$

for all $k = 1, \ldots, m-1$ with $i_0 = i_{m-1}$. Consider an outcome where all agents along the cycle are autarkic:

$$\Lambda_2 = \{\, \{i_1 i_1\}, \{i_2 i_2\}, \ldots, \{i_{m-1} i_{m-1}\} \,\}\,.$$

Notice that all relevant no-blocking conditions IR and PS are satisfied with respect to these agents. Consider for example agent i_k. Agent i_k prefers to be linked with agent i_{k+1}. Agent i_{k+1}, however, prefers to be autarkic than to be in a matching with i_k. Thus a multilateral outcome in which all agents along the cycle are autarkic as in Λ_2 and all other agents are linked following the algorithm presented in Case I constitutes a stable outcome.

This completes the proof of Case II.

Only if: Let $\Delta = \Omega \cup \Gamma \cup \Sigma(\Gamma)$ be a feasible activity structure and let $\mathcal{U}_L$ be the collection of hedonic utility profiles exhibiting link-based externalities. We show by contradiction the necessity of the condition that Γ contains no cycles or if it contains a cycle it is a cycle with a number of connected agents equal $m \geqslant 5$ with $m-1 \neq 3s$ with $s \in \mathbb{N}$.
Let there be a stable multilateral outcome in the network economy (N, Δ, u) for all $u \in \mathcal{U}_L$. Let the network structure Γ contain a cycle $C = (i_1, i_2, \ldots, i_m\}$ with $i_k, i_{k+1} \in \Gamma$ for all $k = 1, \ldots, m-1$ and $m \geqslant 4$ and $m-1 \neq 3s$ with $s \in \mathbb{N}$ with $s > 1$. We discuss two cases: when the cycle consists of exactly three connected agent and when the number of agents is *not* a multiple of 3.
Case I: Suppose first that the cycle contains exactly 3 agents, $C = (i_1, i_2, i_3, i_1)$ and consider the utility profile exhibiting link-based externalities with $\delta = 2/3$: $u_{i_k}(i_k i_k) = 0$, $u_{i_k}(i_k j) = -\infty$ for all $k = 1, 2, 3$ and for all $j \in N_{i_k}(\Gamma) \setminus \{i_{k-1}, i_{k+1}\}$ with $k_0 = 3$ and $k_4 = 1$; $u_{i_1}(i_1 i_2) = u_{i_2}(i_2 i_3) = u_{i_3}(i_2 i_3) = 2$, $u_{i_1}(i_1 i_3) = u_{i_2}(i_1 i_2) = 1$ and $u_{i_3}(i_1 i_3) = -4$. Using equation (13), we can easily calculate $u_{i_1}(i_1 i_2 i_3) = u_{i_2}(i_2 i_1 i_3) = 3$, $u_{i_1}(i_2 i_1 i_3) = u_{i_2}(i_3 i_1 i_2) = 8/3$, $u_{i_1}(i_3 i_1 i_2) = u_{i_2}(i_1 i_2 i_3) = 7/3$, $u_{i_3}(i_3 i_1 i_2) = -2$, $u_{i_3}(i_1 i_2 i_3) = -8/3$, and $u_{i_3}(i_2 i_1 i_3) = -2/3$.

For the sake of argument suppose a stable multilateral outcome exists. Clearly, in the stable multilateral outcome no agent on the cycle can be linked to an agent not on the cycle as such an outcome would violate the IR condition for the agents on the cycle. We are going to discuss all possible states for agent i_1—autarky, bilateral interactions, and multilateral interactions—and show that in each case at least one of the no-blocking conditions is violated. First, it is clear that we cannot have agent i_1 autarkic as this will violate (a) the PS condition if agent i_2 is also autarkic ($u_{i_1}(i_1 i_1) = 0 < 2 = u_{i_1}(i_1 i_2)$ and $u_{i_2}(i_2 i_2) = 0 < 1 = u_{i_2}(i_1 i_2)$); or (b) the PS* condition if agent i_2 is in a bilateral interaction with i_3 and can add the link with i_1 without deleting the link with i_3 ($u_{i_1}(i_1 i_1) = 0 < 8/3 = u_{i_1}(i_2 i_1 i_3)$ and $u_{i_2}(i_2 i_3) = 2 < 3 = u_{i_2}(i_2 i_1 i_3)$). Next, consider the bilateral interactions for agent i_1, (a) $i_1 i_2$ and (b) $i_1 i_3$: they cannot be part of the stable outcome because in the case of (a) the PS condition for agents i_2 and i_3 is violated ($u_{i_2}(i_1 i_2) = 1 < 2 = u_{i_2}(i_2 i_3)$ and $u_{i_3}(i_3 i_3) = 0 < 2 = u_{i_3}(i_2 i_3)$); and in the case of (b) the IR condition for agent i_3 is violated ($u_{i_3}(i_3 i_3) = 0 > -4 = u_{i_3}(i_1 i_3)$). Thus the only alternative left to discuss is the multilateral outcome when agents on the cycle are linked in a multilateral interaction. In all three possible multilateral interactions, however, the IR condition for agent i_3 is not satisfied. Hence with the above link-based externality profile we have established a contradiction to the statement than stable multilateral outcome exists.

Case II: Next, suppose that the cycle contains a number of agents which is not a multiple of 3. Now, consider a utility profile $u \in \mathcal{U}_L$ such that $u_{i_k}(i_k j) < u_{i_k}(i_k i_k) < u_{i_k}(i_{k-1} i_k) < u_{i_k}(i_k i_{k+1}) < u_{i_k}(i_k i_{k-1} i_{k+1})$ for all $k = 1, \dots, m-1$ with $i_0 = i_{m-1}$ and all $j \in N_{i_k}(\Gamma) \setminus \{i_{k-1}, i_{k+1}\}$. Let $\Lambda^\star$ be a stable multilateral outcome in this network economy. Note that in the stable outcome $\Lambda^\star$ the largest number of agents along the cycle that can form a multilateral interaction that satisfies the IR condition is three.
Since the length of the cycle is not a multiple of 3, in any outcome along the cycle there must be at least one agent who is autarkic or at least two distinct agents who are in a bilateral interaction. We discuss these two sub-cases separately and in both cases establish a contradiction to the statement that $\Lambda^\star$ is stable.

Case II.a: First, suppose that $i_k i_k \in \Lambda^\star$ for some $k = 1, \dots, m-1$. Since $\Lambda^\star$ is a stable outcome, the IR condition is satisfied for all agents in N. Hence, agent i_{k-1} is in a state of autarky or connected to agent i_{k-2} either in the bilateral interaction $g' = \{i_{k-1} i_{k-2}\}$, or in the multilateral interaction $g'' = \{i_{k-2} i_{k-1} i_{k-3}\}$ with $i_0 = i_{m-1}$, $i_{-1} = i_{m-2}$, and $i_{-2} = i_{m-3}$. In all three cases the PS condition is violated: by the definition of the utility profile we have $u_{i_k}(i_{k-1} i_k) > u_{i_k}(i_k i_k)$ and $u_{i_{k-1}}(i_{k-1} i_k) > u_{i_{k-1}}(g'') = u_{i_{k-1}}(g') > u_{i_{k-1}}(i_{k-1} i_{k-1})$. Since $\Lambda^\star$ is stable, then it cannot be that $\{i_k i_k\} \in \Lambda^\star$ for some $i_k \in C$.

Case II.b: Next, let the bilateral interaction $\{i_{k-1}, i_k\} \in \Lambda^\star$ for some $k = 1, \dots, m-1$ and $k_0 = m-1$. Then, agent i_{k-2} is connected to agent i_{k-3} either in the bilateral interaction $g' = \{i_{k-2} i_{k-3}\}$, or in the multilateral interaction $g'' = \{i_{k-3} i_{k-2} i_{k-4}\}$ with $i_0 = i_{m-1}$, $i_{-1} = i_{m-2}$, $i_{-2} = i_{m-3}$, and $i_{-3} = i_{m-4}$. In all cases the no blocking condition PS* is violated: by the definition of the utility profile we have $u_{i_{k-2}}(i_{k-1} i_{k-2} i_k) > u_{i_{k-2}}(g') = u_{i_{k-2}}(g'')$ and $u_{i_{k-1}}(\{i_{k-1} i_{k-2} i_k\}) > u_{i_{k-1}}(i_{k-1} i_k)$ with $k_{-1} = m-2$.
Hence, when Γ contains a cycle with a number of connected agents equal exactly 3 or not

a multiple of three, there are such utility profiles that exhibit link-based externalities, for which there is no stable multilateral outcome in the network economy.
This completes the proof of Theorem 4.8(ii).

Proof of Theorem 4.8(i)

If: Consider a network economy $\mathbb{E} = (N, \Delta, u)$ such that $u \in \mathcal{U}_N$ exhibits no externalities. We consider two cases: (I) when Γ does not contain any cycle; (II) when Γ contains a cycle with a number of connected agents that is a multiple of 3.

Case I: Suppose that Γ is acyclic. Since strong stability implies stability, the proof of Case I follows the steps in Case I of the proof of Theorem 4.8(iii).

Case II: Suppose that Γ has a cycle $C = (i_1, \dots, i_m)$ with $m \geqslant 4$ and $m - 1 = 3s$ for some $s \in \mathbb{N}$. The case when the length of the cycle is at least 6 has been discussed in the proof of Theorem 4.8(ii). Notice that in that discussion we have not used the fact that $\delta \in (0, 1)$, and, therefore, the result carries through to the case under consideration here, i.e., when $\delta = 0$. So we only focus on the case when the length of the cycle equals exactly 3.

Consider a network structure Γ that contains a cycle with exactly three connected agents $C = (i_1, i_2, i_3, i_1)$ and notice that (12) implies that the analysis in the proof of 4.8(ii), Case II holds in the case when the cycle contains only 3 agents.

Only If: To show that when the number of agents on the cycle is not a multiple of 3, there may be utility profiles exhibiting no externalities for which there is no stable outcome, we refer to discussion of the proof of 4.8(ii), Case II of **Only If**. Notice that in none of the utility profiles discussed there we have used the condition that $\delta \in (0,1)$, hence, the discussion extends trivially to the case when $\delta = 0$.

This concludes the proof of Theorem 4.8(i), implying that we have shown all assertions in Theorem 4.8.

C Proof of Theorem 4.13

Before we present the proof we state the following auxiliary result.

Lemma 4 *Let (N, Δ, u) be a network economy and let Γ be a network structure that contains no cycles. Then there exist at last two distinct agents in N who have exactly one link in Γ.*

Proof. It is easy to show that Lemma 4 also follows immediately from the fact that the network structure Γ contains no cycles and the finite number of agents in N. Suppose there is at most one agent in N who has exactly one link in Γ.[17] Take any agent $i \in N$ and suppose she has two links in Γ with agents j and k, respectively, where $j \neq k$. Since all agents but one have at least two links, agent j or k must have at least two links, too. Suppose, agent j has exactly two links with agents i and l where $l \neq i$ and $l \neq k$, otherwise there is a cycle in Γ. Similarly, agent l must have at least two links in Γ. Suppose agent l has exactly two links with agents j and m where $m \neq j$, $m \neq i$ and $m \neq k$ otherwise there is a cycle in Γ. Following the same logical steps one arrives at the conclusion that the absence of cycles in Γ and the finiteness of the agent set requires that there are at least two agents who have exactly one link. ∎

Proof of Theorem 4.13

Let $\mathbb{E} = (N, \Delta, u)$ be a network economy such that u exhibits synergistic externalities with $\alpha_c > 0$ for all potential middlemen $c \in \mathcal{K}(\Gamma)$. Suppose Γ contains no cycles. Without loss of generality suppose that there is a path in Γ connecting any two distinct agents in N.[18] Next we re-label the argents to form a sequence that abides by the following rules:

1. Agents in the set N are labelled $1, 2, \ldots, N$ such that any agent with label k where $k = 2, 3, \ldots, N$, is connected to exactly one agent in the set $1, \ldots, k-1$. By Lemma 4, there are at least two agents in the set N who have exactly one link in Γ. Suppose these are agents i and j. Thus we can re-label $i = 1$ and $j = N$.

2. The length of the paths from agent 1 to any two consecutive agents in the sequence, $k-1$, k with $k = 2, \ldots, N$, i.e. $|p_{1k-1}|$ and $|p_{1k}|$ cannot differ by more than a unit where the path of the agent with the higher label is at least as long as the one of the agent with the lower label ($|p_{1k-1}| + 1 \geq |p_{1k}|$).

[17] Recall that we have ruled out the trivial case when there are agents who are not linked in Γ. Thus, all agents in N have at least one link in Γ.

[18] In other words we assume that the graph consists of a single component. This assumption goes without loss of generality as should there be more than one components in the graph, the reasoning presented below can be applied to each component separately. Since there is no link that connects individuals from different components, there are no externalities that need to be taken into account in the construction of a stable outcome either.

The proof now proceeds by induction. Suppose there are stable outcomes, Λ_{k-2}, Λ_{k-1} and Λ_k, in the network economies restricted to the first $k-2$, $k-1$ and k agents in the sequence and the links amongst them with $k \geq 3$, i.e., (N^s, Γ^s, u) with $N^s = \{1, \ldots, s\}$ and $\Gamma^s = \{ij \in \Gamma$ such that $i \in N^s$ and $j \in N^s\}$ for $s = k-2, k-1, k$.
Consider the network economy $(N^{k+1}, \Gamma^{k+1}, u)$ restricted to the first $k+1$ agents and the links amongst them, i.e., $N^{k+1} = \{1, \ldots, k+1\}$ and $\Gamma^{k+1} = \{ij \in \Gamma$ such that $i \in N^{k+1}$ and $j \in N^{k+1}\}$. Notice that by the construction of the sequence of agents, Γ^{k+1} differs from Γ^k only by the additional link of agent $k+1$ with exactly one agent in N^k. For ease of exposition, suppose that agent $k+1$ has a link with agent k in Γ^{k+1}; agent k has a link with agent $k-1$ in Γ^k; and agent $k-1$ has a link with agent $k-2$ in Γ^{k-1}. In the discussion below, we point out how this restriction can be relaxed.

Case I: Suppose that under Λ_k agent k can add the link with agent $k+1$ without deleting all her links.[19]
If the utility that agent k can gain from becoming a middleman is at most the utility she would lose from the direct link with $k+1$ ($u_k(kk+1) \leq -\alpha_k$) or agent $k+1$ is as better off autarkic as he is in a multilateral interaction where agent k acts as a middleman with two other participants ($u_{k+1}(k+1k+1) \geq u_{k+1}(kk+1) + \alpha_k$), then $\Lambda_{k+1} = \Lambda_k \cup \{k+1k+1\}$ is a stable outcome in this economy.[20]
If, on the other hand, both agents k and $k+1$ are better-off by adding the interaction ($u_k(kk+1) > -\alpha_k$ and $u_{k+1}(kk+1) + \alpha_k > u_{k+1}(k+1k+1)$, then $\Lambda_{k+1} = \Lambda_k \oplus \{kk+1\}$ is stable where the operator $\oplus$ as defined above signifies that agent k has added the link with $k+1$ to her existing interactions in Λ_k. This is the case because all agents who are linked to k in Λ_k gain α_k in utility due to the size-based externality, thus, these agents would not want to deviate in Λ_{k+1} if they do not want to deviate in Λ_k where their utility is lower and have the same set of potential partners.[21]
Last, consider the case when agent k prefers to sever the link with $k-1$ and join $k+1$ in a bilateral interaction ($u_k(kk-1) < u_k(kk+1) < -\alpha_k$) and $k+1$ is better off in the bilateral interaction with k than in an autarky ($u_{k+1}(k+1k+1) < u_{k+1}(kk+1)$). Then the outcome $\Lambda_{k+1} = \Lambda_{k-1} \cup \{kk+1\}$ is stable. To see that recall that the outcome Λ_{k-1} is stable for all N^{k-1} agents and that by Lemma 3 the only link between the set of agents N^{k-1} and $\{k, k+1\}$ is the one between $k-1$ and k. These two players, however, cannot form a blocking pair as clearly the PS and PS* condition when k acts as a middleman are satisfied given the conditions on the utility function of agent k. The PS* condition when $k-1$ acts as a middleman must be satisfied since $\{k-1, k\} \in \Lambda_k$. This implies that either agent $k-1$ under Λ_{k-1} cannot act as a middleman, or that $u_{k-1}(k-1, k) < -\alpha_{k-1}$, hence agent $k-1$ does not want to add the link with k without severing all his existing links in Λ_{k-1}.[22]

[19]By construction this implies that the interaction $\{k-1k\} \in \Lambda_k$.

[20]Notice that here the assumption that agents $k-1$and k have only a link with agent $k-2$ in Γ^{k-1} and $k-1$ in Γ^k, respectively, goes without loss of generality. The same reasoning would hold if k is a middleman of a multilateral interaction with s members and the only amendment that would be necessary is to require that agent $k+1$ is as better off autarkic as in a multilateral interaction with k as a middleman and s other members ($u_{k+1}(k+1k+1) \geq u_{k+1}(kk+1) + \alpha_k s$).

[21]Notice again that the reasoning does not hinge on the assumption that k has a link with only one agent in Γ^k. Moreover, additional straightforward requirements on the ordering of the agents in the sequence can ensure that there are no agents with a label preceding that of $k+1$ who have a link with k and who prefer not to be linked to k in Λ_k but prefer to be linked with her in Λ_{k+1}.

[22]Here, the assumption that agent k has only one link and that is with agent $k-1$ who is preceding her in the sequence requires a clarification. Had agent k have also links with other agents whose labels follow k

Case II: Next, suppose that under Λ_k agent k cannot add the link with agent $k+1$ without deleting all her links. If agent k is at least as better off under the outcome Λ_k as she is in a bilateral interaction with $k+1$ ($u_k(\Lambda_k) \geq u_k(k, k+1)$) or if agent $k+1$ is at least as better-off autarkic than as he is in a bilateral interaction with agent k ($u_{k+1}(k+1, k+1) \geq u_{k+1}(k, k+1)$, then $\Lambda_{k+1} = \Lambda_k \cup \{k+1, k+1\}$ is a stable outcome in this economy. This is easy to see, since by construction agent $k+1$ has a link only with agent k and these two agents do not want to engage,[23] then the stability of Λ_k implies the stability of Λ_{k+1}.

Suppose, instead, that agent k prefers to sever her links in Λ_k to be in a bilateral interaction with $k+1$ ($u_k(\Lambda_k) < u_k(k, k+1)$ and $k+1$ prefers to be in a bilateral interaction with k than autarkic ($u_{k+1}(kk+1) < u_{k+1}(k+1, k+1)$. If agent $k-1$ is at least well-off under outcome Λ_{k-1} as in a multilateral interaction of size 3 with agent k acting as a middleman ($u_{k-1}(\Lambda_{k-1}) \geq u_{k-1}(k-1, k) + \alpha_k$) or the utility agent k gains from the direct link with agent $k-1$ is at most equal to the negative of the size-based externality she can generate as a middleman ($u_k(k-1, k) \leq -\alpha_k$), then $\Lambda_{k+1} = \Lambda_{k-1} \cup \{k, k+1\}$ is a stable outcome. That there are no blocking possibilities between $k-1$ and k is ensured by the stability of Λ_{k-1} and Λ_k, where agent k is either autarkic or in a multilateral interaction of which she is not the middleman (due to the fact that she has to sever all links in Λ_k to add a link with $k+1$), and the above restrictions on the utility profiles which dictate the satisfaction of all non-blocking conditions between players $k-1$ and k.[24]

Last consider the case where agent k prefers to sever her links in Λ_k to be in a bilateral interaction with $k+1$ ($u_k(\Lambda_k) < u_k(k, k+1)$ and $k+1$ prefers to be in a bilateral interaction with k than autarkic ($u_{k+1}(k, k+1) < u_{k+1}(k+1, k+1)$. In addition, let agent $k-1$ be better-off in a multilateral interaction of size 3 with agent k acting as a middleman than under Λ_{k-1} ($u_{k-1}(\Lambda_{k-1}) < u_{k-1}(k-1, k) + \alpha_k$) and the utility agent k gains from adding agent $k-1$ to the multilateral interaction is strictly positive ($u_k(k-1, k) > -\alpha_k$), then $\Lambda_{k+1} = \Lambda_{k-2} \cup \{k.k-1.k+1\}$ is a stable outcome. To see that notice that the only blocking possibility for $k+1$ is to the autarkic state which is ruled out by the preference profile and the fact that $k+1$ gains from the positive size-based externality when $k-1$ joins the multilateral interaction. The same analysis holds for the blocking possibility of agent k, which is ruled out by the preference profile specified above and that Λ_k is stable, thus, the IR is satisfied for all agents, including k.[25] In addition to the autarkic state, which is ruled out as a blocking possibility in a similar fashion as it is done for agents k and $k+1$, agent

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and precede $k+1$, then those agents would have been left autarkic in the stable outcome Λ_{k+1}. Recall that by the definition of the sequence all such agents would be equidistant from the origin, agent 1, as agent $k+1$, thus, all such agents would have had only one link in Γ^{k+1} and that would have been with agent k. Since k would sever all links to be in a bilateral interaction with $k+1$, those agents would remain autarkic with no potential partners to form links but k.

[23]Notice that in this case the assumption that agents $k-1$ and k have only one link in Γ^{k-1} and Γ^k, respectively, goes without loss of generality as no other agent who has a link with k can have a link with $k+1$ by Lemma 3.

[24]Similar to the discussion in footnote 21, had agent k have multiple links with agents whose labels follow hers, those agents would be autarkic in the stable outcome Λ_{k+1}.

[25]If there were other agents but $k+1$ who followed k and had a link with her, the construction of the stable outcome would have involved the addition to the multilateral interaction of all those agents who preferred to be members of the multilateral interaction than being autarkic and who earn sufficiently high utility to k for her to add the link. The remainder of the agents would stay autarkic in Λ_{k+1}. Such an outcome would be stable as neither the autarkic players nor those in the multilateral interaction whose label is higher than k have any other links in Γ^{k+1} but the one with k. As in the analysis provided in the main text, in this case, too, the only blocking possibility for k would be to sever all links but given the utility profile and the fact that Λ_k is stable, the IR condition is satisfied.

$k-1$ may have a blocking possibility with agent $k-2$ due to the failure of the PS or PS* condition when $k-2$ acts as a middleman.[26] These two conditions, however, are guaranteed by the requirement that $u_{k-1}(\Lambda_{k-1}) < u_{k-1}(k-1,k)+\alpha_k$. Therefore, by the above discussion and the fact that Λ_{k-2} is a stable outcome, we have shown that Λ_{k+1} constitutes a stable outcome, too.

Finally to complete the proof by induction we show that the initial conditions for $k = 1, 2, 3$ are satisfied. The case when $k = 1$ is trivial as $\Lambda_1 = |11|$ is clearly stable. Consider $N^2 = \{1, 2\}$ with $\Gamma^2 = \{12\}$. If $u_1(11) \geq u_1(12)$ or $u_2(22) \geq u_2(12)$, then $\Lambda_2 = \{\{11\}, \{22\}\}$ is stable. Otherwise, if both agents prefer to be in a bilateral interaction than autarkic, then $\Lambda_2 = \{12\}$ is stable. Last, consider $N^3 = \{1, 2, 3\}$ with $\Gamma^2 = \{\{12\}, \{23\}\}$. If all agents are at least as better-off in autarky as in any bilateral interaction, $u_1(11) \geq u_1(12)$ or $u_2(22) \geq u_2(12)$ and $u_2(22) \geq u_2(23)$ or $u_3(33) \geq u_3(23)$, then $\Lambda_3 = \{\{11\}, \{22\}, \{33\}\}$ is stable. In case at least one pair of agents who have a link prefer to be in a bilateral interaction than in autarky but the third agent prefers autarky than to be in a multilateral interaction, or the agent who may act as a middleman prefers not to add the link, we have the following stable outcome. If $u_1(11) < u_1(12)$ and $u_2(22) < u_2(12)$ and $u_3(33) \geq u_3(23) + \alpha_2$ or $u_2(23) \leq -\alpha_2$, then $\Lambda_3 = \{\{12\}, \{33\}\}$ is stable. Similarly, if $u_3(33) < u_3(23)$ and $u_2(22) < u_2(23)$ and $u_1(11) \geq u_1(12) + \alpha_2$ or $u_2(12) \leq -\alpha_2$, then $\Lambda_3 = \{\{11\}, \{23\}\}$ is stable. Finally, a multilateral interaction $\Lambda_3 = \{213\}$ would be stable in the following two cases: $u_1(11) < u_1(12)$ and $u_2(22) < u_2(12)$ and $u_3(33) < u_3(23) + \alpha_2$ and $u_2(23) > -\alpha_2$ or $u_3(33) < u_3(23)$ and $u_2(22) < u_2(23)$ and $u_1(11) < u_1(12) + \alpha_2$ and $u_2(12) > -\alpha_2$.

This completes the proof of Theorem 4.13.

[26]The assumption that agent $k-1$ has only a link with $k-2$ in Γ^{k-1} can be relaxed in a similar fashion as the assumption concerning agent k. If there are agents who have labels higher than $k-1$ (other than k) and who have a link with $k-1$, then when $k-1$ severs his links with them, they may only participate in activities with other agents who are equidistant from the origin as $k+1$ (i.e. be in a bilateral interaction or a middleman of a multilateral interaction) given the rules by which the sequence is constructed or be autarkic. Notice that due to Lemma 3 the presence of $k+1$ does not present any further blocking possibilities for such agents than the ones present under Λ_k. Thus Λ_{k+1} can be augmented by including these agents in stable partial outcome of autarkies, bilateral, or multilateral interactions amongst them. Moreover, these agents do not have any links with agents in N^{k-2}, thus the stability of Λ^{k-2} holds through.