Partial Cooperation in Strategic Multi-sided Decision Situations*

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Abstract

We consider a normal form game in which there is a single exogenously given coalition of cooperating players that can write a binding agreement on pre-selected actions. The actions representing other dimensions of the strategy space remain under the sovereign, individual control of the players.

We consider a standard extension of the Nash equilibrium concept denoted as a partial cooperative equilibrium as well as an equilibrium concept in which the coalition of cooperators has a leadership position. Existence results are stated and we identify conditions under which the various equilibrium concepts are equivalent.

We apply this framework to existing models of multi-market oligopolies and international pollution abatement. In a multi-market oligopoly typically a merger paradox emerges in the partial cooperative equilibrium. The paradox vanishes if the cartel attains a leadership position. For international pollution abatement treaties, cooperation by a sufficiently large group of countries results in a Pareto improvement over the standard tragedy of the commons outcome described by the Nash equilibrium.

Keywords: Non-cooperative game; partial cooperation; Partial Cooperative Equilibrium; Leadership Equilibrium; multi-market oligopoly; international pollution abatement.

JEL codes: C72, D43, D71, Q54

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1 Introduction

We consider a game theoretic framework for understanding collaborative decision situations embedded in competitive environments. Often these kinds of agreements on a specific economic issue involve a single group of decision makers who cooperate. This coalition writes a binding agreement concerning certain exogenously given jointly decided actions, while the non-collaborative decision makers remain uninvolved. Moreover, while decision makers subject to this binding agreement may cooperate on certain issues, these same players may act non-cooperatively when interacting with all other players along other dimensions. Examples where these tools are applicable range from areas as diverse as environmental agreements (Barrett 1990, Carraro and Siniscalco 1993), R&D collaborations (Yi and Shin 2000, Banal-Español et al. 2013), and financial alliances between banks (White 1996, Popov and Ongena 2011, In 't Veld and van Lelyveld 2014).

A rigorous analysis of these games with *partial cooperation* calls for the development of specifically tailored equilibrium concepts. Our notion of *partial cooperative equilibrium* extends standard Nash best response rationality to our framework and is similar to the coalition equilibrium concept of Ichiishi (1981). Here, the cooperating players write a binding agreement with regard to a specific set of actions, and, *simultaneously*, act non-cooperatively with regard to their individualistic or "private" strategy. The non-cooperators act independently from the cooperators and select a standard best response strategy to all other players' actions. By supplementing the strategy space of the cooperators with a private strategy, this definition generalises the concept of a partial cooperative game in Chakrabarti et al. (2011). Our partial cooperative equilibrium existence result extends the existence theorems seminally stated in Mallozzi and Tijs (2008a, 2009, 2012) for more restricted environments. Our result is based on techniques seminally developed by Glicksberg (1952).

Next, we consider a *leadership equilibrium* concept, which postulates that the cooperating coalition has a (Stackelberg) leadership position. Hence, after a binding agreement has been signed by the cooperative players, all players – cooperators as well as non-cooperators – make independent decisions with regard to all other actions.

The underlying sequential decision process in a normal form game was seminally discussed and developed in Section 3.5 of Ray (2007). This structure was first implemented to partial cooperative games by Mallozzi and Tijs (2008a) and subsequently extended in Mallozzi and Tijs (2008b), Chakrabarti et al. (2011) and Mallozzi and Tijs (2012). Our leadership equilibrium notion builds on this work. Mallozzi and Tijs (2008a) proposed this concept for the class of symmetric potential games. In a subsequent study, Mallozzi and Tijs (2009) consider symmetric aggregative games. The situation when the cooperating agents are faced with multiple Nash equilibria when interacting with the non-cooperative agents is discussed in Mallozzi and Tijs (2008b). Chakrabarti et al. (2011) extended this further to arbitrary non-cooperative games.

We design an extension to the leadership equilibrium notion along several dimensions. First, we extend the strategy space for the cooperators such that actions not subject to coalitional decision making are decided simultaneously by all players – cooperators as well as non-cooperators. Second, we consider a generalised aggregation of the payoffs of the cooperators to evaluate coalitional decisions. A commonly applied aggregator is the *utilitarian* aggregator. The utilitarian aggregator is imposed, for example, in the theory of (standard) partial cooperative games as seminally developed in Mallozzi and Tijs (2009). Alternative aggregators that can be handled by our generalised frame-

work are the *Rawlsian* aggregator (Rawls 1999) and the *Nashian* aggregator (Nash 1950). Third, following Chakrabarti et al. (2011), we assume that in the case of multiple Nash equilibria, the cooperators act optimistically and employ a max-max strategy with regard to their payoffs. Finally, we study the existence properties of our notions in a much larger class of normal-form games.

We investigate two applications of our general framework. In the first application, we borrow the model of cartel formation in a multi-market Cournot oligopoly from Billand et al. (2014). Firms compete in two separate markets. Cooperators collude with regard to quantity choice in one market and not in the other. In the partial cooperative equilibrium, cooperators are worse off than in the standard competitive Cournot-Nash equilibrium due to the *merger paradox* (Salant et al. 1983) if the market on which cooperation occurs is sufficiently large relative to the competitive market. If the market on which firms cooperate is sufficiently small, however, it may be the case that partial cooperation leads to strictly higher payoffs to all firms compared to the standard competitive equilibrium. On the other hand, in the leadership equilibrium, cartel members as well as regular competitors are better off relative to the Nash equilibrium, achieving a strict Pareto improvement, thereby restoring a clear incentive to establish a cartel in such a multi-market oligopoly.

In the second application, we consider the effects of international pollution abatement treaties. Such treaties are best described as partial cooperative agreements: A single coalition of treaty countries writes a binding agreement on certain aspects of the spectrum of economic controls at the disposal of a country's government. The 1997 Kyoto treaty, for example, only regulated emissions of carbon dioxide ("carbon production") rather than the carbon consumption, which normally results from many other economic variables (Helm 2012, Newell et al. 2013).

Our partial cooperative approach supplements Chander and Tulkens (1997)'s model of emissions control with an additional labour input besides the usual polluting factor. Cooperators choose the amount of labour freely, but form a cooperative agreement with regard to the amount of pollutants. In this context, there is no difference between the partial cooperative equilibrium and the leadership one, but both differ from the Nash outcome. Here, only cooperators reduce the amount of pollutants and the level of reduction is greater, the larger is the number of cooperators. All non-treaty countries act as free-riders in these equilibrium situations.

2 Generalised partial cooperative games

We consider normal form games in which an ex-ante postulated group of players collaborates and writes binding agreements on a given subset of actions. All players outside this coalition of cooperators are assumed to follow their individual objectives.

Throughout, let $C = \{1, \ldots, k\}$ be a given and pre-determined *coalition of cooperators*, where $k \geqslant 2$. Furthermore, let $N = \{1, \ldots, n\}$ with $n \geqslant 1$ be the set of non-cooperative players. Thus, the population of all players is given by $C \cup N$, consisting of k cooperating players and n individualistic players. We indicate a generic member of the coalition of cooperators by $i \in C$, while a generic non-cooperative player is denoted by $j \in N$. We consider three separate types of actions:

- Each individual $i \in C$ controls the selection of a **private action** $x_i \in X_i$, where $X_i \neq \varnothing$ denotes i's private action set. We let $X = \prod_{i \in C} X_i$ be the private action tuple set of the coalition of cooperators C.

- The coalition of cooperators C selects cooperatively a **collective action** $y \in Y \neq \emptyset$.[1]
- Finally, each non-cooperator $j \in N$ selects an **individual action** $z_j \in Z_j \neq \emptyset$. We denote $Z = \prod_{j \in N} Z_j$ the non-empty action tuple set of the group of non-cooperators N.

Given the three types of actions we now denote by

$$a = (x, y, z) \in A \equiv X \times Y \times Z \tag{1}$$

a general *action tuple*, where A is the space of all feasible action tuples.

Additionally, following accepted conventions, for every cooperator $i \in C$ and private action tuple $x \in X$ we denote by $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ the actions assigned to all other cooperators in C. Similarly, for every non-cooperator $j \in N$ and action tuple $z \in Z$ we denote by $z_{-j} = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$ the actions assigned to all other non-cooperators in N.

Moreover, each cooperating player $i \in C$ is endowed with a payoff function $u_i \colon A \to \mathbb{R}$ and non-cooperator $j \in N$ is similarly endowed with a payoff function $v_j \colon A \to \mathbb{R}$. We denote by $u = (u_1, \ldots, u_k)$ the list of payoff functions over all cooperators and by $v = (v_1, \ldots, v_n)$ the list of payoff functions for all non-cooperators.

The collective decisions of the coalition of cooperators C are guided by some aggregation of the payoffs of its members (Negishi 1963). Formally, we introduce an *aggregator* as a function $\Lambda \colon \mathbb{R}^k \to \mathbb{R}$ such that the coalition of cooperators C evaluates each action tuple $a = (x, y, z) \in A = X \times Y \times Z$ through the aggregated payoff function $U \colon A \to \mathbb{R}$ defined by

$$U(a) = U(x, y, z) = \Lambda\left(u_1(x, y, z), \ldots, u_k(x, y, z)\right). \tag{2}$$

We say that the aggregator Λ is *Paretian* if it is non-decreasing in u_i, $i \in C$. Throughout, we refer to two common Paretian aggregators. The *Rawlsian* aggregator is given by $\Lambda^r(u_1, \ldots, u_k) = \min\{u_1, \ldots, u_k\}$, while the *utilitarian* aggregator is defined as $\Lambda^u(u_1, \ldots, u_k) = \sum_{i=1}^{k} u_i$.

Clearly, the collective payoff function U is founded on the aggregator Λ and the payoff functions $u_1, \ldots, u_k$ of its members.

Definition 2.1 *A **generalised partial cooperative game** is a list* $\Gamma = \langle C, N, X, Y, Z, u, v, \Lambda \rangle$.

Note here as well that the case where Y is a singleton set reverts any generalised partial cooperative game to a standard $(k + n)$-player normal form game.

2.1 Partial cooperative equilibrium

This equilibrium notion extends Nash's best response rationality to all decisions, namely the individual and private actions of the players as well as the collective action of the coalition of cooperators. The latter evaluates the outcomes of their selection through the aggregated payoff function U based on the aggregator Λ.

[1] This generalises Mallozzi and Tijs (2008a) and Chakrabarti et al. (2011), in which the set of collective actions, Y, is modelled as a k-dimensional vector space with $Y = \prod_{i \in C} Y_i$. In this case, $y_i \in Y_i$ denotes the individual player's $i \in C$ strategic (sub-)action as part of the collective action $y \in Y$. In many applications it is additionally assumed that for every cooperator $i \in C$ the action set Y_i is in fact some subset of a Euclidean space.

Definition 2.2 *An action tuple $a^* = (x^*, y^*, z^*) \in A$ is called a **partial cooperative equilibrium** in $\Gamma = \langle C, N, X, Y, Z, u, v, \Lambda \rangle$ if it satisfies the following conditions:*

(i) *For every cooperating player $i \in C$ it holds that*

$$u_i(x^*, y^*, z^*) \geqslant u_i(x_i, x^*_{-i}, y^*, z^*) \tag{3}$$

for every private action $x_i \in X_i$;

(ii) *For the coalition of cooperators C it holds that*

$$U(x^*, y^*, z^*) \geqslant U(x^*, y, z^*) \tag{4}$$

for every collective action $y \in Y$, where U is given by (2);

(iii) *And for every non-cooperator $j \in N$ it holds that*

$$v_j(x^*, y^*, z^*) \geqslant v_j(x^*, y^*, z_j, z^*_{-j}) \tag{5}$$

for every individual action $z_j \in Z_j$.

Next, we investigate the existence of partial cooperative equilibria in arbitrary generalised partial cooperative games.

Theorem 2.3 *Let $\Gamma = \langle C, N, X, Y, Z, u, v, \Lambda \rangle$ be a generalised partial cooperative game such that*

(i) *All action sets X_i ($i \in C$), Z_j ($j \in N$) and Y are compact and convex subsets of Euclidean spaces;*

(ii) *The payoff function u_i is continuous on $X_i \times Y$, quasi-concave on X_i and concave on Y for every cooperator $i \in C$ and the payoff function v_j is continuous and quasi-concave on Z_j for every non-cooperator $j \in N$,[2] and;*

(iii) *The aggregator Λ is continuous, Paretian and quasi-concave on $\mathbb{R}^k$.*

Then Γ admits at least one partial cooperative equilibrium.

For a proof of Theorem 2.3 we refer to the appendix of this paper.

2.2 Leadership equilibrium

Next we implement a sequential structure on the decision-making process in generalised partial cooperative games. We consider an internal two-tier hierarchical structure in which the coalition of cooperators C has a leadership position in the decision-making process. Thus, in the first instance C writes a binding agreement resulting in the selection of some collective action $y \in Y$ and, subsequently, all cooperators $i \in C$ and all non-cooperating players $j \in N$ respond to that selection by selecting private actions $x_i \in X_i$, respectively $z_j \in Z$.

[2] Let A be convex. A function $f\colon A \to \mathbb{R}$ is *concave* if if for all $\lambda \in (0,1)$ and $a', a'' \in A$: $f(\lambda \cdot a' + (1-\lambda) \cdot a'') \geqslant \lambda \cdot f(a') + (1-\lambda) \cdot f(a'')$, and f is *quasi-concave* if for all $\lambda \in (0,1)$ and $a', a'' \in A$: $f(\lambda \cdot a' + (1-\lambda) \cdot a'') \geqslant \min\{f(a'), f(a'')\}$.

The leadership equilibrium concept reflects a standard backward induction logic in which the coalition of cooperators C acts as a Stackelberg leader. The coalition of cooperators C is assumed to act from an optimistic point of view in this procedure and considers only outcomes from the second-stage interaction between all agents that correspond to the best possible outcome for them collectively.

In order to define the leadership equilibrium concept properly, we have to introduce some hypotheses on the fundamentals.

Axiom 2.4 *Consider a generalised partial cooperative game $\Gamma = \langle C, N, X, Y, Z, u, v, \Lambda \rangle$. We assume that the following properties hold:*

(i) *For every $i \in C$ it holds that X_i is a compact and convex subset of some Euclidean space;*

(ii) *For every $j \in N$ it holds that Z_j is a compact and convex subset of some Euclidean space;*

(iii) *The set of collective actions Y is a compact subset of some Euclidean space;*

(iv) *For every cooperator $i \in C$, the payoff function $u_i : A \to \mathbb{R}$ is a continuous function and the section $u_i(\cdot, x_{-i}, y, z) : X_i \to \mathbb{R}$ is quasi-concave on X_i for all $(x_{-i}, y, z) \in X_{-i} \times Y \times Z$;*

(v) *For every non-cooperator $j \in N$, the payoff function $v_j : a \to \mathbb{R}$ is a continuous function and the section $v_j(x, y, \cdot, z_{-j})$ is quasi-concave on Z_j for all $(x, y, z_{-j}) \in X \times Y \times Z_{-j}$, and;*

(vi) *The aggregator Λ is continuous on $\mathbb{R}^k$.*

The assumptions introduced in Axiom 2.4 are weaker than the ones imposed in Theorem 2.3.

Under Axiom 2.4 we can now formulate some auxiliary notations that are required for the definition of the leadership equilibrium concept. For any given collective action $y \in Y$ we denote by $\Gamma_y = \langle C \cup N, X \times Z, w^y \rangle$ a standard normal form game given by player set $C \cup N$ and payoff functions w_i^y for all players in $C \cup N$ that are described as follows:

- First, for every cooperator $i \in C$ we consider the conditional payoff function $w_i^y : X \times Z \to \mathbb{R}$ given by

$$w_i^y(x, z) = u_i(x, y, z). \tag{6}$$

- Second, for every non-cooperating player $j \in N$ the given action set Z_j as well as a conditional payoff function $w_j^y : X \times Z \to \mathbb{R}$ defined as

$$w_j^y(x, z) = v_j(x, y, z). \tag{7}$$

The non-cooperative normal form game $\Gamma_y = \langle C \cup N, X \times Z, w^y \rangle$ is denoted as the *conditional partial cooperative game* for the collective action $y \in Y$. The set of Nash equilibria of the conditional game Γ_y is now denoted by $E_y \subset X \times Z$. Under the given hypotheses this definition is non-trivial:

Lemma 2.5 *Consider a generalised partial cooperative game $\Gamma = \langle C, N, X, Y, Z, u, v, \Lambda \rangle$ satisfying Axiom 2.4. Then for every collective action $y \in Y$ the set of Nash equilibria E_y of the conditional partial cooperative game $\Gamma_y = \langle C \cup N, X \times Z, w^y \rangle$ forms a non-empty, compact subset of the (Euclidean) strategy set $X \times Z$.*

Lemma 2.5 is an immediate corollary from the more general assertion stated as Lemma A.2 in the appendix of this paper.

We postulate that the cooperators have an *optimistic* outlook on their abilities to gain from their leadership position.[3] Using the definition of U given in (2) and the assumptions made in Axiom 2.4, we let for every $y \in Y$,

$$U^s(y) = \max_{(x,z)\in E_y} U(x,y,z) \tag{8}$$

$$\Phi^s = \arg\max U^s \equiv \left\{ y' \in Y \;\middle|\; U^s(y') = \max_{y\in Y} U^s(y) \right\}. \tag{9}$$

The function U^s assigns the maximum payoff level to the coalition of cooperators C that can be achieved when the cooperators as well as the non-cooperators do a best response in their respective private and individual actions to all players' actions.[4]

The set Φ^s of U^s-maximisers describes the best responses of the coalition of cooperators C to the other players' actions. Indeed, Φ^s is the collection of collective actions that maximise the maximum payoff envelope function U^s over Y, i.e., these collective actions are coalition C's best responses given the collective payoff function U^s.

We can now define the leadership equilibrium concept for Γ as follows:

Definition 2.6 *Let $\Gamma = \langle C, N, X, Y, Z, u, v, \Lambda\rangle$ be a generalised partial cooperative game that satisfies Axiom 2.4. An action tuple $(x^\star, y^\star, z^\star) \in A$ is a **leadership equilibrium** in Γ if $y^\star \in \Phi^s$ such that $E_{y^\star} \neq \varnothing$ and, furthermore, that*

$$(x^\star, z^\star) \in \arg\max\nolimits_{(x,z)\in E_{y^\star}} U(x, y^\star, z). \tag{10}$$

where U is defined as in (2).

The existence of a leadership equilibrium in a partial cooperative game can be established under the assumed conditions.

Theorem 2.7 *Let $\Gamma = \langle C, N, X, Y, Z, u, v\rangle$ be a generalised partial cooperative game that satisfies Axiom 2.4. Then there exists at least one leadership equilibrium in Γ.*

For a proof of Theorem 2.7 we again refer to the appendix.

2.3 Separability

A special class of generalised partial cooperative games is characterised by the separation of decisions concerning, on the one hand, the collective actions (y) and, on the other hand, the private (x) as well as individual (z) actions. Formally, a generalised partial cooperative game Γ =

[3] We remark that this optimistic outlook can be replaced by other approaches such as a minimax logic or even a fully pessimistic outlook.

[4] These concepts are indeed properly defined given the assumptions on u_i, $i \in C$, and Λ stated in Axiom 2.4 and the compactness of E_y from Lemma 2.5.

$\langle C, N, X, Y, Z, u, v, \Lambda \rangle$ is called **separable** if $\Lambda = \Lambda^u$ is the utilitarian aggregator and for every $i \in C$ and $(x, y, z) \in X \times Y \times Z$:

$$u_i(x, y, z) = \hat{u}_i(x, z) + \tilde{u}_i(y) \tag{11}$$

where $\hat{u}_i : X \times Z \to \mathbb{R}$ and $\tilde{u}_i : Y \to \mathbb{R}$ are partial payoff functions.

Separability guarantees that the two notions of equilibrium defined here for the class of generalised partial cooperative games coincide. Indeed, in a separable game the collective decision of coalition C concerning the collective actions y is completely described by the maximisation of $\tilde{U} = \sum_{i \in C} \tilde{u}_i$ and, therefore, is strictly independent of the selection of individual actions of the cooperators $i \in C$. This implies immediately the following assertion.

Proposition 2.8 *Let Γ be a separable generalised partial cooperative game satisfying Axiom 2.4. Then the notion of leadership equilibrium is equivalent to that of the partial cooperative equilibrium concept in Γ in the sense that every partial cooperative equilibrium is a leadership equilibrium.*

In a separable partial cooperative game with multi-dimensional collective actions in the sense that $Y = \prod_{i \in C} Y_i$ for appropriately chosen Y_i, one can also introduce the notion of a standard Nash equilibrium. This notion is rather useful in the analysis of the effects of collective decision making on certain specified actions.

Definition 2.9 *An action tuple (x', y', z') is a **Nash equilibrium** in a separable partial cooperative game Γ with $Y = \prod_{i \in C} Y_i$ if for every cooperator $i \in C$ the pair (x'_i, y'_i) maximises $u_i(\cdot, x'_{-i}, \cdot, y'_{-i}, z')$ over $X_i \times Y_i$ and for every non-cooperator $j \in N$ the individual action z'_j maximises $v_j(x', y', \cdot, z'_{-j})$ over Z_j.*

In general, the class of Nash equilibria is different from the class of Partial Cooperative and Leadership equilibria in these separable games. This is shown through the following example, which considers a separable generalised partial cooperative game with two collaborators.

Example 2.10 Consider a separable partial cooperative game Γ with $C = \{1, 2\}$ and N arbitrary. We only specify the payoffs resulting from the collective actions and let

$$\tilde{u}_1(y_1, y_2) = -(y_1 - 1)^2 + 2y_2 - 3$$
$$\tilde{u}_2(y_1, y_2) = 2y_1 - 3 - (y_2 - 1)^2$$

Then in the Nash equilibrium for this game we have that $y'_1 = y'_2 = 1$ regardless of the payoff functions and the strategies of all other players. On the other hand, in the partial cooperative equilibrium (y_1, y_2) maximises

$$\tilde{U}(y_1, y_2) = \tilde{u}_1(y_1, y_2) + \tilde{u}_2(y_1, y_2) = -(y_1 - 2)^2 - (y_2 - 2)^2.$$

This implies that in the PCE we arrive at $y^*_1 = y^*_2 = 2$, again regardless of the actions of all other players in the game. ♦

We refine the notion of separability further to arrive at some further conclusions concerning the equivalence of Nash equilibria and the other equilibrium concepts. Specifically, a generalised partial

cooperative game $\Gamma = \langle C, N, X, Y, Z, u, v, \Lambda \rangle$ is called **homogeneous** if Γ is separable such that $Y = \prod_{i \in C} Y_i$ is multi-dimensional and for every cooperator $i \in C$ and every action tuple $(x, y, z) \in X \times Y \times Z$:

$$u_i(x, y, z) = \hat{u}_i(x, z) + \tilde{u}(y) \tag{12}$$

where $\hat{u}_i \colon X \times Z \to \mathbb{R}$ and $\tilde{u} \colon Y \to \mathbb{R}$.

We can now state the following assertion without proof. A proof is based on the insight that the determination of the collective action $y \in Y = \prod_{i \in C} Y_i$ is not only completely separated from the determination of the other actions $(x, z) \in X \times Z$, but that it is essentially based on the maximisation of the common payoff function $\tilde{u}$. Thus, the optimal collective action coincides with the individually optimal choice of $y_i \in Y_i$ for every cooperator $i \in C$.

Proposition 2.11 *Let Γ be a homogeneous generalised partial cooperative game. Then the notion of partial cooperative equilibrium is equivalent to that of Nash equilibrium in Γ in the sense that every partial cooperative equilibrium is a Nash equilibrium.*

3 Two applications

In this section we investigate two applications of generalised partial cooperative games that clearly delineate the various equilibrium concepts. Both of these applications consider a generalised partial cooperative game formulation that allows the development of a standard Nash equilibrium. Hence, the concept of a standard Nash equilibrium is well-defined and can be computed. Thus, a complete comparison of these Nash equilibria with the two main partial cooperative equilibrium concepts—partial cooperative equilibrium and leadership equilibrium—can be developed.

3.1 Cartels in multi-market oligopolies

We first consider a cartel in a multi-market oligopoly. Firms are fully competitive in one market, while a subset of firms forms a cartel in the second market only. We show that the cartel benefits in comparison with the standard Nash equilibrium outcome if it has a clear leadership position: An improvement only occurs in the leadership equilibrium, not in the partial cooperative equilibrium.

We limit our analysis to the case of three firms indexed by $i = 1, 2, 3$ that compete in two related markets A and B.[5] We assume that these firms engage in quantity-setting, Cournotian competition. We denote by q_i and Q_i the quantities sold by firm i on markets A and B, respectively. Furthermore, p and P represent the market prices emerging in market A and B, respectively.

Focusing on the most interesting case, we suppose that competitors' products are strategic substitutes and there are diseconomies of scope across the two markets. More specifically, demand in markets A and B for firm i are respectively represented by

$$p = \alpha - \sum_{i=1}^{m} q_i \quad \text{and} \quad P = \beta - \sum_{i=1}^{m} Q_i$$

[5] Throughout we use the notation of Billand et al. (2014).

where $\alpha, \beta > 0$ are demand parameters describing the total market size for each of the two goods A and B.

We assume a well-established, widely adopted production technology imposing a simple, identical quadratic cost function on the three firms, given by

$$C(q, Q) = \tfrac{1}{2}(q + Q)^2 \tag{13}$$

Therefore, the profit of firm i is determined as

$$\pi_i = p \cdot q_i + P \cdot Q_i - C(q_i, Q_i) = p \cdot q_i + P \cdot Q_i - \tfrac{1}{2}(q + Q)^2. \tag{14}$$

It can be shown that in this linear quadratic formulation, the resulting second order conditions are always satisfied if α and β are not too different. Hence, one obtains a unique interior maximum through the first order conditions.

Formulation as a generalised partial cooperative game. We set up a generalised partial cooperative game that describes cartel formation in this multi-market Cournot oligopoly. We consider the formation of the cartel $C = \{1, 2\}$ only in market B, while full competition is retained in market A. Here, firm 3 remains an independent producer.[6]

In this multi-market framework, $y = (Q_1, Q_2)$ forms the *collective* action of the cartel $C = \{1, 2\}$, while quantities (q_1, q_2) set in the A-market constitute *private* actions for the two cartel members $i = 1, 2$. Firm 3 acts completely independently in both markets and set all output levels (q_3, Q_3) as *individual* actions.

The generalised partial cooperative game is completed with the selection of the utilitarian aggregator $\Lambda^u(\pi_1, \pi_2) = \pi_1 + \pi_2$ to direct the collective decisions of the cartel $C = \{1, 2\}$.[7]

Equilibrium analysis

In this setting, we discuss the Nash equilibrium, the partial cooperative equilibrium and the leadership equilibrium in this model of cartel formation in a multi-market oligopoly.

We take Nash equilibrium as a benchmark with respect to which we compare the results of the two partial cooperative analysis. If all three firms act competitively and (q_i, Q_i), $i = 1, 2, 3$, are set independently, then a straightforward analysis shows that a unique symmetric Nash equilibrium exists if $\frac{\alpha}{5} < \beta < 5\alpha$. This equilibrium is given by

$$q_1^{NE} = q_2^{NE} = q_3^{NE} = \frac{5\alpha - \beta}{24} \equiv q^{NE}$$

$$Q_1^{NE} = Q_2^{NE} = Q_3^{NE} = \frac{5\beta - \alpha}{24} \equiv Q^{NE}.$$

[6] We remark that our analysis can be extended to an arbitrary number of competitors and cartel members. This generalization, however, leads to rather cumbersome expressions that add rather little to the conclusions from our analysis. The case for any number of competitors is available upon request from the authors.

[7] We mention here that replacing the utilitarians aggregator by the Rawlsian aggregator $\Lambda^r = \min\{\pi_1, \pi_2\}$ would not affect the analysis of this particular application. In particular, this is due to the symmetry of the described situation.

In the symmetric Nash equilibria all firms earn equal profits:

$$\pi_1^{NE} = \pi_2^{NE} = \pi_3^{NE} = \frac{17\alpha^2 + 17\beta^2 - 2\alpha\beta}{288} \equiv \pi^{NE}$$

Partial cooperative and leadership equilibria: Next, we investigate the partial cooperative and leadership equilibria in this generalised partial cooperative game.

Proposition 3.1 *Given a multi-sided oligopoly with firm profit functions given by* (14)*, then*

- *A Partial cooperative equilibrium exists and is unique if* $\frac{\alpha}{5} < \beta < 5\alpha$ *and is given by*

$$q_1^{PE} = q_2^{PE} = \frac{20\alpha - 2\beta}{98}; \quad q_3^{PE} = \frac{21\alpha - 7\beta}{98};$$

$$Q_1^{PE} = Q_2^{PE} = \frac{15\beta - 3\alpha}{98}; \quad Q_3^{PE} = \frac{25\beta - 5\alpha}{98};$$

- *A unique leadership equilibrium exists and is given by*

$$q_1^{LE} = q_2^{LE} = \frac{205\alpha - 37\beta}{968}; \quad q_3^{LE} = \frac{197\alpha - 45\beta}{968};$$

$$Q_1^{LE} = Q_2^{LE} = \frac{193\beta - 49\alpha}{968}; \quad Q_3^{LE} = \frac{19\beta - 3\alpha}{88}.$$

Deriving the above results involves straightforward albeit tedious calculations and thus are not presented here.[8] We compute that

$$\pi_1^{PE} = \pi_2^{PE} = \frac{1125\alpha^2 + 1077\beta^2 - 78\alpha\beta}{19208} \quad \text{and} \quad \pi_3^{PE} = \frac{1188\alpha^2 + 1672\beta^2 - 512\alpha\beta}{19208}$$

for the partial equilibrium case, and

$$\pi_1^{LE} = \pi_2^{LE} = \frac{229\alpha^2 + 229\beta^2 - 26\alpha\beta}{3872} \quad \text{and} \quad \pi_3^{LE} = \frac{26673\alpha^2 + 29557\beta^2 - 2314\alpha\beta}{468512}.$$

for the leadership equilibrium.

For $i = 1, 2$ we conclude that

$$\pi_i^{LE} \geqslant \max\left\{\pi_i^{PE}, \pi_i^{NE}\right\}. \tag{15}$$

We can now also investigate whether the *merger paradox*, seminally stated for Cournot competition in Salant et al. (1983) emerges in any of these environments. The merger paradox would occur if cartel formation in this multi-market setting results into losses to its members. Clearly, (15) implies that there are no grounds for merger paradox to emerge if the cartel obtains a leadership position in the multi-market oligopoly, and, that instead there may be market conditions that make it a feature of the partial cooperative equilibrium when $\pi_i^{PE} < \pi_i^{NE}$ for the cartel members $i = 1, 2$. Here, that is the case for $\beta > \frac{317}{409}\alpha \approx 0.775\alpha$. Thus, the merger paradox occurs if market B is sufficiently larger than the competitive market A. On the other hand, forming a cartel on a relatively smaller

[8] Detailed derivations are available from the authors upon request.

market may result in a strict Pareto improvement. We compute that this is particularly the case for $\frac{1951}{3875}\alpha < \beta < \frac{317}{409}\alpha$.

We can illustrate this point more clearly by presenting the equilibrium outcome for specific values of the market parameters. For example, if $\alpha = 5$ and $\beta = 3$, we arrive at $\pi_1^{PE} = \pi_2^{PE} = 1.9080 > \pi^{NE} = 1.9028$ as well as $\pi_3^{PE} = 1.9298 > \pi^{NE} = 1.9028$. Here, a leadership role improves the profit position for the cartel, although the independent producer firm 3 still outperforms its competitors. To be precise, we compute that $\pi_1^{LE} = \pi_2^{LE} = 1.9101 > \pi_1^{PE} = \pi_2^{PE} > \pi^{NE}$, while $\pi_3^{PE} > \pi_3^{LE} = 1.9170 > \pi^{NE}$.

3.2 International pollution abatement

Our second application concerns an international environmental protection situation in which certain countries write an international pollution abatement treaty, while other countries do not participate in such a collective control of their emissions. Our analysis builds on the seminal contribution by Chander and Tulkens (1997) who for the purpose of their analysis developed the notion of the γ-core for partition function form games. We remark here that their notion of *partial agreement equilibrium* corresponds to our notion of partial cooperative equilibrium.

We amend their analysis by supplementing the cooperators' strategy space with an additional dimension along which binding agreements and cooperation are not possible. So, we include an input factor—denoted as *labour*—that treaty countries choose independently, similarly to non-cooperators.

There are two types of countries, namely those that cooperate on writing an international emissions control treaty—denoted by $C = \{1, \ldots, k\}$ with $k \geqslant 2$—and those that act independently—denoted by $N = \{1, \ldots, n\}$ with $n \geqslant 1$. All countries produce two non-tradable goods, 1 and 2, only. Treaty country $i \in C$ produces and consumes quantities q_{1i} and q_{2i}, while non-treaty country $j \in N$ produces and consumes q_{1j} and q_{2j}, respectively.

Good 1 is "clean" in the sense that its production leaves no emissions footprint. Good 2 is "dirty" in the sense that its production requires an input that leaves an emissions footprint. For all countries the production technology is the same. For generic country $h \in C \cup N$ the production functions for the two goods are given by $q_{1h} = \sqrt{L_{1h}}$ and $q_{2h} = \sqrt{d_h L_{2h}}$, where $L_{1h} \geqslant 0$ and $L_{2h} \geqslant 0$ are, respectively, the amounts of labour expended in the production of goods 1 and 2 and $d_h \geqslant 0$ is the "dirty input" in the production of good 2.

We further assume that every country $h \in C \cup N$ is endowed with the same total labour input equal to $\overline{L}$. Thus, $L_{1h} + L_{2h} \leq \overline{L}$. Throughout we assume that $\overline{L} > 1$.

The utility function of a representative consumer of country $h \in C \cup N$ is now given by

$$u_h(q_{1h}, q_{2h}, \Delta) = q_{1h} + q_{2h} - \tfrac{1}{2}\Delta \tag{16}$$

with

$$\Delta = \sum_{i=1}^{k} d_i + \sum_{j=1}^{n} d_j \tag{17}$$

$$q_{1h} = \sqrt{L_{1h}} \tag{18}$$

$$q_{2h} = \sqrt{d_h L_{2h}} \tag{19}$$

$$L_{1h} + L_{2h} \leqslant \overline{L} \tag{20}$$

Here, Δ represents the emissions associated with the production of good 2 across the global community.[9] Note that this payoff function is *separable* in the production variables (q_{1h}, q_{2h}) on the one hand and the emission decisions $\{d_h\}_{h\in C\cup N}$ on the other. This implies that the generalised partial cooperative game considered here is separable and Proposition 2.8 applies.

In terms of the notation employed, $y = (d_1, \ldots, d_k)$ is the multi-dimensional collective action of all treaty countries in C, representing the treaty written between the countries in C to control emissions. Throughout, we postulate that the coalition of treaty countries C is guided by utilitarian principles and uses the utilitarian aggregator $\Lambda^u = \sum_{i\in C} u_i$ to guide its collective decisions.

Equilibrium analysis

We can approach this model as a normal form strategic game as well as a generalised partial cooperative game. We first determine its Nash equilibrium that we use as a benchmark. This represents a situation in which all countries in C act independently and do not write a treaty to control their emissions. This results into a standard tragedy of the commons problem which given by the maximisation of the objective function (16) subject to the constraints listed in (17)–(20).

Indeed, in the Nash equilibrium for each country $h \in C \cup N$ we derive that $L_{1h}^{NE} = 1$ and $L_{2h}^{NE} = d_h^{NE} = \overline{L} - 1$, resulting into $u_h^{NE} = \overline{L} - \frac{1}{2}(k+n)(\overline{L}-1)$.[10]

Partial cooperative and leadership equilibria: First, note that the formulated generalised partial cooperative game is separable and that Proposition 2.8 now implies that the partial cooperative equilibria are equal to the leadership equilibria in this model. We, therefore, limit our discussion to the partial cooperative equilibria only.

In the case of cooperation, the coalition of treaty countries C signs a binding agreement $y = (d_1, \ldots, d_k)$ and each country $h \in C\cup N$ chooses independently the amount of labour that it allocates to the production of each of the two goods (L_{1h}, L_{2h}). Moreover, each non-treaty country $j \in N$ makes a decision on the dirty input d_j.

The coalition of treaty countries C determines $y = (d_1, \ldots, d_k)$ by solving to the following

[9] To bring to the fore the mechanism of partial cooperation, we take countries to be completely homogeneous and goods to be non-tradable. As a consequence, all countries remain in a state of autarky.

[10] Obviously, $u^{NE} < 0$ if the total number of countries $k+n \geqslant 3$ $(\geqslant 4)$ and the total in-country labour input $\overline{L} > 3$ (> 2). In that case, the negative impact of pollution overtakes the positive utility from the consumption of the two goods.

maximisation problem:

$$\max_{d_1,\ldots,d_k} \sum_{i\in C}\left(\sqrt{\overline{L}-L_{2i}}+\sqrt{d_i L_{2i}}-\frac{1}{2}\left(\sum_{i=1}^{k} d_i+\sum_{j=1}^{n} d_j\right)\right)$$

This yields that for every $i \in C$: $d_i = \dfrac{L_{2i}}{k^2}$. This now results in the following conclusions:[11]

Proposition 3.2 *There are two cases for which a complete description of the partial cooperative equilibrium of the pollution abatement model can be determined:*

(a) $1 < \overline{L} \leqslant k^2$:
For each treaty country $i \in C$: $L_{1i}^{PE} = \overline{L}$ *and* $L_{2i}^{PE} = d_i^{PE} = 0$. *For each independent country* $i \in N$: $L_{1j}^{PE} = 1$ *and* $L_{2j}^{PE} = d_j^{PE} = \overline{L} - 1$.
The total pollution in this equilibrium is determined as $\Delta^{PE} = n(\overline{L}-1)$ *and the resulting utility levels as*

$$u_i^{PE} = \sqrt{\overline{L}} - \tfrac{n}{2}(\overline{L}-1)$$

$$u_j^{PE} = \left(1-\tfrac{n}{2}\right)\overline{L} + \tfrac{n}{2}$$

For all $1 < \overline{L} \leqslant k^2$, *the partial cooperative equilibrium is a strict Pareto improvement over the Nash equilibrium.*

(b) $\overline{L} > k^2$:
For each treaty country $i \in C$: $L_{1i}^{PE} = k^2$, $L_{2i}^{PE} = \overline{L} - k^2$ *and* $d_i^{PE} = \frac{\overline{L}-k^2}{k^2}$. *For each independent country* $i \in N$: $L_{1j}^{PE} = 1$ *and* $L_{2j}^{PE} = d_j^{PE} = \overline{L} - 1$.
The total pollution in this equilibrium is determined as

$$\Delta^{PE} = \left(n+\tfrac{1}{k}\right)\overline{L} - (k+n)$$

and the resulting utility levels as

$$u_i^{PE} = \frac{\overline{L}-k^2}{k} + \tfrac{1}{2}(3k+n) - \tfrac{1}{2}\left(n+\tfrac{1}{k}\right)\overline{L}$$

$$u_j^{PE} = \overline{L} - \tfrac{1}{2}\left(n+\tfrac{1}{k}\right)\overline{L} + \tfrac{1}{2}(k+n)$$

For any $\overline{L} > k^2$ *the partial cooperative equilibrium is a strict Pareto improvement over the Nash equilibrium.*

The conclusion from Proposition 3.2 is that treaty countries as well as independent countries are better-off under pollution abatement treaties than under global non-collaboration as described in the Nash equilibrium. Although in Nash equilibrium, all countries achieve the same utility levels, under partial cooperation, the independent countries have a higher utility than the cooperators, since they act as free-riders, enjoying the environmental pollution abatement imposed through the

[11] The computations and derivations are rather tedious and therefore relegated to the appendix of this paper.

treaty. As expected, total pollution always increases with the number of independent countries. An increase in the number of treaty countries, on the other hand, decreases total pollution.

4 Concluding remarks

In this paper we introduce two equilibrium notions that are applicable to situations of partial cooperation. We establish their existence properties in a general and widely applicable setting.

First, as our concept of partial cooperative equilibrium builds on the standard notion of Nash equilibrium, it is neutral to the assumption of observability of players' strategies, including the existence and composition of the coalition of cooperators.

Second, even though our results are derived in the context of a single group of cooperators, our results on the partial cooperative equilibrium concept are generalisable to a setting with an arbitrary number of exclusive coalitions of cooperators. For a partial cooperative equilibrium to exist in such more general settings, the conditions on the aggregator function identified in Theorems 2.3 must apply to all coalitions of cooperators.

In the context of leadership in partial cooperation, we present our results under the assumption that the cooperators have a leadership role. A similar existence result can be derived if, instead, one of the non-cooperators acts as the Stackelberg leader. To see that, it is sufficient to recognise that once the first mover takes action, the players that are moving next are *de facto* in a situation of partial cooperation, where the existence of a fixed point is established in Theorem 2.3.

Finally, we point out the role of complete information in our set up. Here as in the standard Nash Equilibrium analysis we take complete information to implicitly imply common knowledge of payoffs. In the context of leadership equilibrium, we make an implicit requirement that the collective action is observable by all players, too.

Possible extensions of our analysis. Despite its robustness, we acknowledge that our analysis is restrictive in the sense that cooperation between the coalition members is *ex-ante* postulated and binding. In this respect, our work is complementary to the large literature on the endogenous formation of agreements based on the work of Ichiishi (1981), Bloch (1997), and Yi (1997).[12] Whereas the focus of this literature is on the existence and stability of a stable coalition structure given a payoff allocation, our contribution is on the equilibrium derivation of payoffs in a strategic environment taking the coalition structure as given. While in these contexts the existence of equilibria is not an issue, our contribution guides future research on the entire class of games where equilibrium existence is guaranteed by deriving very general conditions on the payoff functions. Clearly, further effort is needed in the direction of studying the general properties of the simultaneous game of coalition formation and partial cooperation.

In the context of the two specific applications developed here, an indication of coalition stability can be derived from the comparison between the partial equilibrium payoffs and those in the Nash Equilibrium. If one takes Nash equilibrium to be resulting when cooperation breaks down, then conditions under which partial (leadership) equilibrium payoffs for the cooperators are higher are suggestive of the range of conditions necessary to sustain cooperative agreements.

[12] For an overview of this literature, we refer to Finus and Rundshagen (2009).

The two applications discussed also underline both the significance and the relevance of the equilibrium notions based on partial cooperation. In particular, in our first application we embed a standard framework of multi-market oligopolistic competition into our analysis of partial cooperation. By doing so, we generalise the analysis in existing studies—e.g. Bernheim and Whinston (1990) and more recently Choi and Gerlach (2013)—that analyse the profitability of cartels in multi-market context. In this literature, cartels are of limited size and occur in the absence of outside competitive pressures. Our analysis suggests that when firms meet on two markets, cartels are more likely to be stable on the relatively smaller market, or where the cartel can take a leadership position relative to non-cartel members. To the best of our knowledge we are the first to point out the advantages that cartels attain in multi-market oligopolies in the context of sequential decision making.

Our second application concerning environmental agreements draws out the role of separability in the analysis. As discussed in the introduction, partial cooperation in this context has been studied since the 1990s. The main conclusion of our analysis—that payoffs for the signatories are equal under partial cooperative equilibrium and under leadership equilibrium—point to the fact that arguments that a success of partial cooperation is dependent on taking a first-mover advantage are not always justifiable.

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A Appendix: Proofs of theorems and propositions

This appendix collects the proofs of the main existence theorems presented in Section 2 of this paper.

A.1 Proof of Theorem 2.3

For every cooperator $i \in C$, we define her best response correspondence as a mapping $\gamma_i : X_{-i} \times Y \times Z \to 2^{X_i}$ given by

$$\gamma_i(x_{-i}, y, z) = \arg\max_{x_i \in X_i} u_i(x_i, x_{-i}, y, z) \tag{21}$$

for any $(x_{-i}, y, z) \in X_{-i} \times Y \times Z$.

Similarly, for the coalition of cooperators C itself, we define the collective best response correspondence $\delta : X \times Z \to 2^Y$ by

$$\delta(x, z) = \arg\max_{y \in Y} U(x, y, z) \tag{22}$$

for any $(x, z) \in X \times Z$, where U is the aggregated payoff function for C defined in (2).

Furthermore, for every non-cooperator $j \in N$, we define her best response correspondence $\varepsilon_j : X \times Y \times Z_{-j} \to 2^{Z_j}$ by

$$\varepsilon_j(x, y, z_{-j}) = \arg\max_{z_j \in Z_j} v_j(x, y, z_j, z_{-j}) \tag{23}$$

for any $(x, y, z_{-j}) \in X \times Y \times Z_{-j}$.

This allows us to introduce the joint best response correspondence $\mathbf{B} : A \to 2^A$ such that for any $\hat{a} = (\hat{x}, \hat{y}, \hat{z}) \in A$,

$$\mathbf{B}(\hat{a}) = \left\{ (x, y, z) \in A \;\middle|\; \begin{array}{l} x_i \in \gamma_i(\hat{x}_{-i}, \hat{y}, \hat{z}) \text{ for all } i \in C, \\ y \in \delta(\hat{x}, \hat{z}), \text{ and} \\ z_j \in \varepsilon_j(\hat{x}, \hat{y}, \hat{z}_{-j}) \text{ for all } j \in N \end{array} \right\}. \tag{24}$$

It is clear that a fixed point of this best response correspondence—defined as some $a^\star \in A$ such that $a^\star \in \mathbf{B}(a^\star)$—corresponds to a partial cooperative equilibrium of the generalised partial cooperative game Γ.

We proceed by showing that $\mathbf{B}$ indeed possesses such a fixed point.
First, we show that the best response correspondences γ_i ($i \in C$), δ, and ε_j ($j \in N$) are all non-empty valued.
Given that X_i is compact and u_i continuous on X_i, for every $i \in C$, applying the Weierstrass Theorem implies that u_i indeed admits a maximum and, thus, $\gamma_i(x_{-i}, y, z) \neq \varnothing$ for all $a = (x, y, z) \in A$ where $i \in C$.
Next, since the aggregator Λ is continuous and all u_i, $i \in C$, are continuous on Y, it follows that U is continuous on Y as well. From compactness of Y, it follows that δ is therefore non-empty valued.
Finally, the compactness of Z_j and continuity of v_j on Z_j, for all $j \in N$, implies that ε_j is non-empty valued as well for all $j \in N$.
Therefore, combining these facts it follows that $\mathbf{B}$ is a non-empty valued correspondence.

Next, we show that $\mathbf{B}$ is convex valued.
First we claim that each of the correspondences γ_i ($i \in C$), δ, and ε_j ($j \in N$) are convex-valued. To see this, consider γ_i for some $i \in C$. For any $(x_{-i}, y, z) \in X_{-i} \times Y \times Z$, $\gamma_i(x_{-i}, y, z)$ is the set of maxima of the quasi-concave function $u_i(\cdot, x_{-i}, y, z)$ mapped onto the convex set X_i. Hence, $\gamma_i(x_{-i}, y, z)$ is

indeed a convex set.
Similar arguments can be used to show that for any $(x, y, z_{-j}) \in X \times Y \times Z_{-j}$, the set $\varepsilon_j(x, y, z_{-j})$ is convex.
Finally, consider δ. Since Λ is Paretian and quasi-concave, it aggregates quasi-concave utility functions in a quasi-concave function.[13] Therefore, U is quasi-concave on Y.
Furthermore, for every $(x, z) \in X \times Z$, $\delta(x, z)$ is the set of maxima of the quasi-concave function $U(x, \cdot, z)$ mapped on a convex set Y, implying that $\delta(x, z)$ is indeed convex.
This implies that $\mathbf{B}$ is indeed convex valued.

Finally, we prove $\mathbf{B}$ is upper hemi-continuous.
Consider a sequence $\hat{a}_p = (\hat{x}_p, \hat{y}_p, \hat{z}_p)$ converging to some $\hat{a} = (\hat{x}, \hat{y}, \hat{z})$, as well as a sequence $\tilde{a}_p = (\tilde{x}_p, \tilde{y}_p, \tilde{z}_p)$ converging to some $\tilde{a} = (\tilde{x}, \tilde{y}, \tilde{z})$ such that $(\tilde{x}_p, \tilde{y}_p, \tilde{z}_p) \in \mathbf{B}(\hat{x}_p, \hat{y}_p, \hat{z}_p)$ for all $p \in \mathbb{N}$. We now prove that $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbf{B}(\hat{x}, \hat{y}, \hat{z})$, implying that the correspondence $\mathbf{B}$ is closed and, thus, since A is compact, it follows that $\mathbf{B}$ is indeed upper hemi-continuous.
By definition it follows for all $i \in C$ that for every $x_i \in X_i$:

$$u_i(\tilde{x}_{p,i}, \hat{x}_{p,-i}, \hat{y}_p, \hat{z}_p) - u_i(x_i, \hat{x}_{p,-i}, \hat{y}_p, \hat{z}_p) \geqslant 0 \tag{25}$$

For every $y \in Y$:

$$U(\hat{x}_p, \tilde{y}_p, \hat{z}_p) - U(\hat{x}_p, y, \hat{z}_p) \geqslant 0 \tag{26}$$

Finally, for every $j \in N$ and all $z_j \in Z_j$:

$$v_j(\hat{x}_p, \hat{y}_p, \tilde{z}_{p,j}, \hat{z}_{p,-j}) - v_j(\hat{x}_p, \hat{y}_p, z_j, \hat{z}_{p,-j}) \geqslant 0 \tag{27}$$

From these conclusions it follows immediately that

$$u_i(\tilde{x}_i, \hat{x}_{-i}, \hat{y}, \hat{z}) - u_i(x_i, \hat{x}_{-i}, \hat{y}, \hat{z}) \geqslant 0 \text{ for every } i \in C \text{ and } x_i \in X_i; \tag{28}$$

$$U(\hat{x}, \tilde{y}, \hat{z}) - U(\hat{x}, y, \hat{z}) \geqslant 0 \quad \text{for every } y \in Y; \tag{29}$$

$$v_j(\hat{x}, \hat{y}, \tilde{z}_j, \hat{z}_{-j}) - v_j(\hat{x}, \hat{y}, z_j, \hat{z}_{-j}) \geqslant 0 \text{ for every } j \in N \text{ and } z_j \in Z_j \tag{30}$$

This proves that $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbf{B}(\hat{x}, \hat{y}, \hat{z})$.

Hence, we conclude that the correspondence $\mathbf{B}\colon A \to 2^A$ is a convex-valued and upper hemi-continuous correspondence. From Kakutani's fixed point theorem (Border 1985, page 72), it can be concluded that $\mathbf{B}$ admits a fixed point and, hence, a partial cooperative equilibrium exists for the generalised partial cooperative game Γ.

A.2 Proof of Theorem 2.7

The proof of Theorem 2.7 is based on Berge's Theorem. For completeness we state this fundamental result here.

Lemma A.1 ***(Berge 1997, page 115)***
Let M and N be two Euclidean spaces and let $D\colon M \to 2^N$ be a non-empty and compact valued correspondence and $f\colon M \times N \to \mathbb{R}$ be a upper semi-continuous function. Define the value function $V\colon M \to \mathbb{R}$ by

$$V(m) = \sup\{f(n, m) \mid n \in D(m)\}.$$

If D is upper hemi-continuous at $m \in M$, then the value function V is upper semi-continuous at m.

[13] We refer to Negishi (1963, Proposition 1) for a concise proof of this result.

Next consider a generalised partial cooperative game $\Gamma = \langle C, N, X, Y, Z, u, v, \Lambda\rangle$. First we define the Nash correspondence which is equal to the feasibility correspondence E defined in Section 2 of the paper. Formally, we introduce the mapping $E\colon Y \to 2^{X\times Z}$ by $E(y) = E_y$ for every collective action $y \in Y$. We show that under standard assumptions, this correspondence is non-empty and compact valued as well as upper hemi-continuous.

Lemma A.2 *Let for the generalised partial cooperative game $\Gamma = \langle C, N, X, Y, Z, u, v\rangle$, X_i be a compact and convex subset of some Euclidean space for every cooperator $i \in C$ and Z_j be a compact and convex subset of some Euclidean space for every non-cooperator $j \in N$. Assume that X, Y, and Z are all non-empty.*
Let for all $i \in C$, $u_i\colon A \to \mathbb{R}$ be a continuous function and $u_i(\cdot, x_{-i}, y, z)\colon X_i \to \mathbb{R}$ be quasi-concave for all $(x_{-i}, y, z) \in X_{-i} \times Y \times Z$. Furthermore, let for all $j \in N$, $v_j\colon A \to \mathbb{R}$ be a continuous function and $v_j(x, y, \cdot, z_{-j})$ be quasi-concave for all $(x, y, z_{-j}) \in X \times Y \times Z_{-j}$.
Then the correspondence $E\colon Y \to 2^{X\times Z}$ is non-empty and compact valued as well as upper hemi-continuous.

Proof. For an arbitrary collective action $y \in Y$, consider the reduced normal form game $\Gamma_y = \langle C \cup N, X \times Z, w\rangle$ as given before. Assume that X_i is a compact and convex subset of a Euclidean space for each $i \in C$ and Z_j is a compact and convex subset of a Euclidean space for all $j \in N$.
For any $i \in C$, the function w_i^y is equal to a section of u_i and, hence, by assumption its section $w_i^y(\cdot, x_{-i}, z)$ is quasi-concave on X_i.
For any $j \in N$, the function w_j^y is equal to a section of v_j and, consequentially, by assumption its section $w_j^y(x, \cdot, z_{-j})$ is quasi-concave on Z_j.
Moreover, both w_i^y ($i \in C$) and w_j^y ($j \in N$) are continuous on $X \times Z$.
Hence, we conclude that the game $\Gamma_y = \langle C \cup N, X \times Z, w\rangle$ satisfies all conditions of a standard Nash equilibrium existence theorem (Fudenberg and Tirole 1991, page 34) and, therefore, admits a Nash equilibrium. Thus, $E_y \neq \varnothing$ for all $y \in Y$ implying that $E\colon Y \to 2^{X\times Z}$ is non-empty valued.

Next, we show that the correspondence E is closed, and since $X \times Z$ is compact, E is upper hemi-continuous as well. Consider a convergent sequence $\hat{y}_p \to \hat{y}$ and a convergent sequence $(\hat{x}_p, \hat{z}_p) \to (\hat{x}, \hat{z})$ such that $(\hat{x}_p, \hat{z}_p) \in E(\hat{y}_p)$. We prove that $(\hat{x}, \hat{z}) \in E(\hat{y})$.
Indeed, by definition, for all $i \in C$, we have $w_i^y(\hat{x}_{p,i}, \hat{x}_{p,-i}, \hat{z}_p) - w_i^y(x_i, \hat{x}_{p,-i}, \hat{z}_p) \geqslant 0$ for all $x_i \in X_i$. Similarly, for all $j \in N$, $w_j^y(\hat{x}_p, \hat{z}_{p,j}, \hat{z}_{p,-j}) - w_j^y(\hat{x}_p, z_j, \hat{z}_{p,-j}) \geqslant 0$ for all $z_j \in Z_j$. It follows immediately that both $w_i^y(\cdot,\cdot,\cdot) - w_i^y(x_i,\cdot,\cdot)\colon X_i \times X_{-i} \times Z \to \mathbb{R}$ and $w_j^y(\cdot,\cdot,\cdot) - w_j^y(\cdot, z_j, \cdot)\colon X \times Z_j \times Z_{-j} \to \mathbb{R}$ are continuous functions. Hence, for every $i \in C$, we have that $w_i^y(\hat{x}_i, \hat{x}_{-i}, \hat{z}) - w_i^y(x_i, \hat{x}_{-i}, \hat{z}) \geqslant 0$ and, for all $j \in N$, we have $w_j^y(\hat{x}, \hat{z}_j, \hat{z}_{-j}) - w_j^y(\hat{x}, z_j, \hat{z}_{-j}) \geqslant 0$. This implies $(\hat{x}, \hat{z}) \in E(\hat{y})$. Thus, E is indeed a closed correspondence.

Finally, take any $y \in Y$. We show that $E(y)$ is compact. Since E is a closed correspondence, it is, therefore, closed-valued. Hence, $E(y)$ is a closed subset of a compact set $X \times Z$ and, thus, $E(y)$ is compact. ∎

Next, consider the function $U^s\colon Y \to \mathbb{R}$ as defined in (8). From the continuity of Λ and u_i for each $i \in C$, we conclude that U is continuous on A. Since, from Lemma A.2, the set E_y is compact for all $y \in Y$, it then follows that $\arg\max_{(x,y)\in E_y} U(x, y, z)$ exists for every $y \in Y$. Hence, U^s is a well-defined function. It remains to show that U^s is an upper semi-continuous function.

Lemma A.3 *Let the generalised partial cooperative game $\Gamma = \langle C, N, X, Y, Z, u, v, \Lambda\rangle$ satisfy the requirements of Theorem 2.7. Then the indirect utility function U^s is upper semi-continuous.*

Proof. In Lemma A.1, we make the substitutions, $M = Y$, $N = X \times Z$, $D = E$ and $f = U$. We have proven in Lemma A.2 that E is indeed upper hemi-continuous as well as non-empty and compact valued. Furthermore, by construction, the utility function U is continuous and, thus, $f = U$ is upper semi-continuous.

Finally, we can see that from Berge's formulation that $V = U^s$ by construction. By applying Lemma A.1, we conclude that U^s is upper semi-continuous. ■

We are now in the position to complete the proof of Theorem 2.7.
From Lemma A.3, $U^s : Y \to \mathbb{R}$ is an upper semi-continuous function defined on a compact set Y. Hence, it follows from standard results that Φ^s is non-empty.
Take any $y^* \in \Phi^s$. From Lemma A.2, E_{y^*} is non-empty as well as compact. Moreover, $U(\cdot, y^*, \cdot) : X \times Z \to \mathbb{R}$ is a continuous function. Hence, from the Weierstrass Theorem,

$$\Psi = \arg\max_{(x,z)\in E_{y^*}} U(x, y^*, z) \neq \varnothing.$$

Take any $(x^*, z^*) \in \Psi$. Then it is easy to establish that (x^*, y^*, z^*) constitutes a leadership equilibrium in the generalised partial cooperative game Γ.
This completes the proof of Theorem 2.7.

A.3 Proof of Proposition 3.2

We proceed by straightforwardly compute the first order conditions from the optimisation problems for the treaty countries $i \in C$, the independent countries $j \in N$ and for C as a collective. First, notice that without loss of generality, we can replace the inequality in (20) by an equality and re-write the constrained maximisation problem given by Equations (16)-(20) as a straightforward maximisation of a single function with two variables given by

$$\max_{L_{2h}, d_h} u_h = \sqrt{\overline{L} - L_{2h}} + \sqrt{d_h L_{2h}} - \tfrac{1}{2}\left(\sum_{i=1}^{k} d_i + \sum_{j=1}^{n} d_j\right). \tag{31}$$

The first order conditions of (31) are given by:

$$\frac{\partial u_h}{\partial L_{2h}} = -\frac{1}{2\sqrt{\overline{L} - L_{2h}}} + \frac{d_h}{2\sqrt{d_h L_{2h}}} \equiv 0 \tag{32}$$

$$\frac{\partial u_h}{\partial d_h} = \frac{L_{2h}}{2\sqrt{d_h L_{2h}}} - \tfrac{1}{2} \equiv 0. \tag{33}$$

We re-call that we already established that for the treaty countries $i \in C$, the equilibrium level of individual country emissions is given by

$$d_i = \frac{L_{2i}}{k^2},$$

which through substitution in (32) implies that

$$\frac{1}{2\sqrt{\overline{L} - L_{2i}}} = \tfrac{1}{2k}$$

We first show assertion (b) for $\overline{L} > k^2$. Hence, $L_{2i}^{PE} = \overline{L} - k^2 > 0$ and $d_i^{PE} = \frac{\overline{L} - k^2}{k^2}$. In turn, we then establish that $L_{1i}^{PE} = \overline{L} - L_{2i} = k^2$.
The first order conditions for the independent countries $j \in N$ now imply the computed outcomes for the Nash equilibrium, i.e., $L_{1j}^{PE} = 1$ and $L_{2j}^{PE} = d_j^{PE} = \overline{L} - 1$.

Therefore,

$$\Delta^{PE} = k\left(\frac{\overline{L}-k^2}{k^2}\right) + n\left(\overline{L}-1\right) = (n+\tfrac{1}{k})\overline{L} - (k+n).$$

This implies now that

$$u_i^{PE} = k + \frac{\overline{L}-k^2}{k} - \tfrac{1}{2}(n+\tfrac{1}{k})\overline{L} + \tfrac{1}{2}(k+n) =$$
$$= \tfrac{1}{2}(k+n) - \tfrac{1}{2}\left(n-\tfrac{1}{k}\right)\overline{L}$$

$$u_j^{PE} = \overline{L} - \tfrac{1}{2}\left(n+\tfrac{1}{k}\right)\overline{L} + \tfrac{1}{2}(k+n).$$

Now since $\Delta^{PE} < \Delta^{NE}$ we derive immediately that for every $j \in N$: $u_j^{PE} > u_j^{NE}$. Furthermore, for $i \in C$ it holds that $u_i^{PE} > u_i^{NE}$ if and only if

$$\frac{\overline{L}-k^2}{k} + \tfrac{1}{2}(3k+n) - \tfrac{1}{2}(n+\tfrac{1}{k})\overline{L} > \overline{L} - \tfrac{1}{2}(k+n)\overline{L} + \tfrac{1}{2}(k+n)$$

if and only if

$$\overline{L} > \left(1 - \tfrac{k}{2} + \tfrac{1}{2k}\right) k\,\overline{L}.$$

Obviously, $\overline{L} > 0$, so the inequality simplifies to

$$\left(1 - \tfrac{k}{2} + \tfrac{1}{2k}\right) k < 1 \quad \text{or} \quad (k-1)^2 > 0.$$

This is obviously the case for any value of k in the assumed range $k \geqslant 2$, showing assertion (b).

Next we show assertion (a) for $1 < \overline{L} \leqslant k^2$.
In that case there is a corner solution for the equilibrium conditions described by $d_i^{PE} = L_{2i}^{PE} = 0$ and $L_{1i}^{PE} = \overline{L}$ for the treaty countries $i \in C$.
For the independent countries $j \in N$ we derive again $L_{1j}^{PE} = 1$ and $L_{2j}^{PE} = d_j^{PE} = \overline{L} - 1$.
Thus, $\Delta^{PE} = n(\overline{L}-1)$ and

$$u_i^{PE} = \sqrt{\overline{L}} - \tfrac{n}{2}(\overline{L}-1) \quad \text{and} \quad u_j^{PE} = \overline{L} - \tfrac{n}{2}(\overline{L}-1) = \left(1-\tfrac{n}{2}\right)\overline{L} + \tfrac{n}{2}$$

To investigate the Pareto domination between the PE and the NE outcomes, we note again that $u_j^{PE} > u_j^{NE}$ as $\overline{L} > 1$ and now $u_i^{PE} > u_i^{NE}$ if and only if

$$\sqrt{\overline{L}} - \tfrac{n}{2}(\overline{L}-1) > \overline{L} - \tfrac{1}{2}(k+n)(\overline{L}-1)$$

if and only if

$$\sqrt{\overline{L}} > \overline{L} - \tfrac{k}{2}\left(\overline{L}-1\right)$$

if and only if

$$2\sqrt{\overline{L}} > k - (k-2)\overline{L}.$$

Now $\overline{L} > 1$ and $k \geqslant 2$ imply that

$$2\sqrt{\overline{L}} > 2 > k - (k-2)\overline{L}.$$

Hence, $u_i^{PE} > u_i^{NE}$, thus confirming the assertion of (a) in Proposition 3.2.