The Provision of Collective Goods through a Social Division of Labour*

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Abstract

We develop a general equilibrium framework in which a wide range of collective economic configurations are provided through specialised professionals as part of an endogenously emerging social division of labour. We extend the theory of value to this setting bringing together a model of an economy with collective goods with the model of a private-goods market economy with an endogenously emerging social division of labour. Natural applications are the presence of non-tradables in production, the effects of education on productive abilities, and the market system itself as an implementation of the price mechanism.

For an appropriately generalised notion of *valuation equilibrium*, we prove the two fundamental theorems of welfare economics under very general conditions, notably allowing for incomplete, non-monotonic, and non-transitive preferences. We also incorporate Adam Smith's principle of increasing returns to specialisation and establish that there emerge well-structured social divisions of labour in equilibrium.

Keywords: Social division of labour; Consumer-producer; Collective goods; Pareto optimality; Valuation equilibrium; The Welfare Theorems.

JEL classification: H41, D41, D51

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1 Value creation through a social division of labour

The study of the provision collective goods in general equilibrium theory has been pursued mainly in a setting of a strict dichotomy between the production and consumption of all goods, private as well as collective. We consider an economy in which individual economic agents are *consumer-producers,* endowed with productive abilities as well as consumptive preferences, as in Yang (1988, 2001). All decisions by these consumer-producers are guided by the same price system for private goods and a personalised tax-subsidy system for the financing of the collective goods.

Our approach is founded on the formation of an endogenous social division of labour to produce all private goods in the economy Gilles (2018b, 2019a,b). In our model, inputs of these private goods can be converted into abstract collective goods through a specified production matrix. The delivery of collective goods is financed through an appropriately devised tax-subsidy system. The private good inputs that the public authority uses, would typically also include specialised human capital through the employment of specific classes of professionals.

Our framework is widely applicabile to numerous issues in the theory of value. More precisely, our model allows proper valuations to be assigned to many aspects of societal organisation that affect the process of generating wealth. Traditionally, collective goods fit this category and are used to represent government-provided public goods or economic sources of widespread externalities. Both of these traditional applications fit well within our general framework.[1] In Section 2 of this paper, we also discuss the valuation of less traditional wealth generating factors such as non-tradable inputs to production processes and knowledge.

Our model builds on the framework of valuation equilibrium in an economy with such unstructured, non-Samuelsonian collective goods developed by Mas-Colell (1980), Diamantaras and Gilles (1996), and Diamantaras, Gilles, and Scotchmer (1996) as well as the model of a competitive economy with an endogenous social division of labour devised by Gilles (2018b, 2019a,b). Our model extends beyond these frameworks by considering economic agents to be endowed with very general consumptive preferences introduced in Hildenbrand (1969) as well as general cost structures related to the provision of these collective goods.

We endow each agent with preferences over consumption bundles of private goods as well as collective good configurations. These preferences are non-satiated, have thin indifferences sets, and satisfy minimal essentiality conditions for the collective goods. Each agent is also endowed with a production set of individually feasible production plans. These individual production sets are affected by external effects of the collective goods that are provided in the economy. This allows, for example, the consideration of public education systems and their effects on the emerging social division of labour.

The provision cost of the collective goods is represented by a correspondence that assigns to every collective good configuration a set of private good inputs. Thus, every configuration of collective goods can be provided via a variety of different input plans.

A continuum of agents is an apt setting in which to develop our model. The social division of

[1] In particular, we refer to the analysis of education and its effects on productivity and the endogenous social division of labour (Bowles, Gintis, and Meyer, 1975) as well as environmental issues such as the adoption of green production technologies through appropriate subsidies (Hanley, McGregor, Swales, and Turner, 2006).

labour is only fully implementable with a large enough population of economic agents. A continuum population allows the economy to generate flexible outputs from such a large collaborative production system. This is fully exploited in the theory set out in Gilles (2019b) as well as in our model.

Our approach is sufficiently general to encompass many of the existing models of collective good provision in general equilibrium theory. This includes private good endowment economies, home production economies, and private ownership production economies. We provide specifications of our framework that show the inclusion of these models.

In our setting, an *allocation* consists of (i) an allocation of private good consumption bundles, (ii) a collective good configuration, (iii) an input configuration for the provision of the selected collective good configuration, and (iv) an assignment of production plans to individual consumer-producers. An allocation is *feasible* if all private good markets clear and there are sufficient amounts of the private goods produced to provide the collective good in its chosen configuration and social provision of the chosen collective good configuration. A feasible allocation is *Pareto optimal* if there is no alternative feasible allocation that is uniformly better for a set of agents of positive measure without being worse for any agent.

Our analysis focusses on Pareto optimal allocations and their support through an appropriate price-valuation system. We generalise the notion of *valuation equilibrium*—developed by Mas-Colell (1980) and Diamantaras and Gilles (1996)—to our setting. A valuation equilibrium is a feasible allocation that is supported through a conditional private good price system and a valuation—a tax-subsidy scheme—such that all consumption and production plans are optimal and that the collective good configuration maximises the budgetary surplus, which in equilibrium is exactly zero.

We establish the following insights in the general theory:

- We show the First Fundamental Theorem of Welfare Economics: Valuation equilibria are Pareto optimal under minimal conditions on the agents' production sets and their consumptive preferences (Theorem 4.6).

- Furthermore, every Pareto optimal allocation can be supported weakly by a private good price system and an appropriately devised tax-subsidy system if production is collectively bounded and all consumptive preferences satisfy an essentiality condition in the sense that for every individual, any collective good configuration can be compensated by sufficient quantities of the private goods (Theorem 4.9).

- We strengthen this insight to a full statement of the Second Fundamental Theorem of Welfare Economics that all Pareto optimal allocations can be supported as valuation equilibria if consumptive preferences are continuous and directionally monotone and satisfy stronger versions of the essentiality condition (Theorem 4.14(a)). Similarly, we show that an irreducibility condition can replace one of these strict essentiality conditions to support Pareto optima as valuation equilibria (Theorem 4.14(b)). These insights show that full support of Pareto optima can be achieved if one assumes sufficiently strong hypotheses on the consumptive preferences, which contrasts sharply with the mild conditions that suffice to establish weaker support of these Pareto optima.

- Finally, we consider explicitly the consequences of Adam Smith's notion of Increasing Returns to Specialisation (IRSpec) as formalised by Gilles (2018b, 2019a,b). IRSpec posits that one's productive abilities improve if one specialises in the production of a single output. We show that under IRspec all economic agents select full specialisation production plans only and there emerges a strict social division of labour in which agents occupy specific professions only (Theorem 5.2(b)). Also, there emerges complete income equality among these specialisations due to the assumed perfect mobility among professions under perfect competitive conditions (Theorem 5.2(c)).

We focus on the support of Pareto optimal allocations as valuation equilibria and the two welfare theorems. The existence of Pareto optima and valuation equilibria lies outside the scope of this paper. The existence question is non-trivial due to the widespread externalities emanating from the collective goods.

Applications. Our general framework can be applied to a very wide array of subjects and issues. We explore four such applications in this paper. First, in section 2.1 we look at a model of the classic public service of policing, in which consumer-producers specialise in the production of either food or policing. The government then selects a valuation system for policing service provision and this system guides the specialisation of the agents and their consumption choices to an efficient allocation.

Second, in section 2.2 collective good configurations can be used to model non-tradable inputs to production processes. We explore a simple economy with a regulated commons. Usage of the commons acts as a non-tradable input for all production processes pursued by individual members of the economy. Our valuation equilibrium concept introduces a method to value and price the usage of the commons, even though it is not a marketed commodity.

Third, in section 2.3 we investigate the effects of collectively generated knowledge on production processes. Such knowledge directly affects productivity of individual consumer-producers and forms a source for economic growth. Again, the valuation equilibrium system allows the proper assignment of value to such collective knowledge and its uses.[2]

Context and related literature on the social division of labour. Adam Smith eloquently described wealth creation through the specialisation of labour in a social division of labour at the foundation of the economy (Smith, 1776, Book I, Chapters I–III). Here, economic wealth generation is directly related to the ideas of *increasing returns to specialisation* and *gains from trade* as the foundation for a social division of labour, an idea at least as old as Plato's "Republic" written circa 380 BCE (Plato, 380 BCE) and further discussed by Mandeville (1714) and Hume (1740, 1748). The notion of the social division of labour was further developed by Babbage (1835) and, most profoundly, by Marx (1867) for the industrial age.

However, these fundamental principles have been relatively neglected in the modern study of general equilibrium in a market economy. Only relatively recently, Yang (1988, 2001, 2003), Yang and

[2]This application contrasts with the role of collective knowledge in a market economy as discussed by Hayek (1937). There knowledge promotes competition and the well-functioning of the market system and the allocation of scarce resources. In our example knowledge is limited to productive knowledge that directly affects the productivity of individual consumer-producers in the economy.

Borland (1991), Yang and Ng (1993), Sun, Yang, and Zhou (2004) and Gilles (2019b) have developed a modern mathematical approach to wealth creation and allocation through a social division of labour, centred around the notion of a consumer-producer as the building block of such a social division of labour. In the models of these works, all production and consumption decisions are made by economic agents endowed with consumptive as well as productive abilities.

Yang (1988, 2001) broke new ground by introducing a mathematical model of consumer-producers in the context of a continuum economy with an endogenous social division of labour. Sun, Yang, and Zhou (2004) established the existence of a general equilibrium founded on increasing returns to scale in productive abilities in which the competitive price mechanism guides these consumer-producers to socially optimal specialisations. Their framework allowed for transaction costs as well.

Gilles (2019b) introduced a mathematical formulation of the economic notion of increasing returns to specialisation in a competitive market economy with private goods. Gilles shows the existence of competitive equilibria, the fundamental welfare theorems, and how the social division of labour can also supplant prices and direct the allocation of resources efficiently. The crucial assumption behind these results is the law of one price, imposed on all the production and trading processes. This was extended in Gilles (2018a), which explores properties of the Core of such an economy, in particular the role of the social division of labor in Edgeworthian barter processes.

This model can simultaneously give an account of general equilibrium price formation, the process of wealth creation itself, as well as the endogenous allocation of the generated wealth. Equilibration under full specialisation happens by endogenous adaptation of the social division of labor rather than the price mechanism.

Economies with a social division as well as collective goods have also been explored in Basile, Gilles, Graziano, and Pesce (2019), which particularly investigates the properties of Core allocations in these economies, building on the results established in Gilles (2018a).

Context and related literature on collective good provision. Concerning "non-Samuelsonian" collective goods, Mas-Colell (1980) established a model of an economy with one private good ("money") and a collective good that is not necessarily measurable on a cardinal scale and for which the agents necessarily have non-monotonic preferences, since the "amount" of the collective good is undefined, referred to as a *collective good*. This model significantly extends the well-established model of "public" good economies developed by Samuelson (1954), which is founded on the hypothesis that those public goods are quantifiable and subject to monotonic preferences.[3]

Mas-Colell (1980)'s main equilibrium concept is that of *valuation equilibrium* and he introduced the hypothesis of essentiality of the private good to establish the second welfare theorem. Diamantaras and Gilles (1996) generalised this framework to an economy with multiple private goods and established that, in order to establish the welfare theorems, a conditional price system has to be implemented that imposes a private-good price vector for each collective good configuration. This framework was subsequently generalised by Graziano (2007), Graziano and Romaniello (2012) and Basile, Graziano, and Pesce (2016).

[3]Samuelsonian public goods are a (very) special case of our collective goods. We sometimes use the locution "non-Samuelsonian collective goods" for brevity, however, instead of the more accurate "not-necessarily Samuelsonian public goods".

Another relevant field of application is the study of causes of economic inequality. The hypothesis that production occurs through a social division of labour allows to compare the effects of imperfections in the mobility between professional classes, causing opportunity inequality (Roemer, 1998). Furthermore, the control of certain (non-tradable) inputs in the production processes can lead to unequal allocation of wealth in the social division of labour (Dow and Reed, 2013). Our framework can be enhanced to use the presence of collective goods as well as modified equilibrium concepts to capture these aspects of the economy.

2 A universal conception of economic value

Economic wealth creation processes are founded on individual productive abilities and on the collective institutional framework in which such wealth creation takes form. These collective institutional features take many different forms, ranging from government provision of traditional public goods (Samuelson, 1954) to the collective determination of costly institutional market institutions affecting the terms of trade as well as the transaction costs (Gilles and Diamantaras, 2003; Diamantaras, Gilles, and Ruys, 2003).[4]

We set out to develop a common valuation concept that measures the effects of these collective institutional settings on the economic wealth creation processes that is as general as possible. This implies that many studies in the literature can be considered special cases of the general framework set out here. Hence, the Lindahl pricing of traditional Samuelsonian public goods as well as conjectural pricing systems evaluating trade in different trade infrastructures are specifications of the concepts developed here.

We explicitly recognise that these measures of economic value have a common basis and are essentially based on two properties. We present this universal conception here and provide a number of illustrative applications showing their functionality.

Economic valuation. There is a distinction between standard economic goods—which are subject to individual property rights—and configurations of collective institutional elements—which we refer to as a *collective good configuration* to distinguish this from standard, quantifiable economic goods. We assume throughout that there are a finite number $\ell \in \mathbb{N}$ of standard goods as well as a set $\mathcal{Z}$ of potential collective good configurations. We assume that each collective good configuration $z \in \mathcal{Z}$ can be implemented through a variety of equivalent provision schemes, represented by a set of input bundles $C(z) \subset \mathbb{R}^{\ell}_{+}$. The implemented provision plan $c \in C(z)$ is selected by the public authority.

We assume that production is individualised. Hence, every agent a has individual productive abilities described by a correspondence $\mathcal{P}_a \colon \mathcal{Z} \to 2^{\mathbb{R}^{\ell}}$ that assigns a production set $\mathcal{P}_a(z) \subset \mathbb{R}^{\ell}$ subject to which collective good configuration z is implemented.[5] Therefore, production is subject

[4] This application particularly relates to the perspective on the role of collective goods in a market economy as put forward in Book V of Smith (1776). There Smith argues that good governance is based on the establishment of a well-functioning market infrastructure. Economic development and growth is achieved through lower market transaction costs. In the current paper we do not debate the role of transaction costs for which we instead refer to Yang and Borland (1991),Yang (2003), and Gilles and Diamantaras (2003).

[5] The standard case of Gilles (2019b) is captured by $\mathcal{P}_a(z) = \mathcal{P}_a(z') = \mathcal{P}_a \subset \mathbb{R}^{\ell}$ for all $z, z' \in \mathcal{Z}$.

to widespread externalities from the implementation of collective good configurations.

The value of standard economic goods is measured through a *conjectural price system* $p\colon \mathcal{Z} \to \mathbb{R}^{\ell}_{+}$, which can interpreted as a measurement of the price of a standard good subject to the particular collective good configuration considered. Note that a production plan $g \in \mathcal{P}_a(z)$ is *optimal* for agent a under the conjectural price $p(z) \geqslant 0$ if

$$p(z) \cdot g = \sup p(z) \cdot \mathcal{P}_a(z) = \sup\{p(z) \cdot g \mid g \in \mathcal{P}_a(z)\}.$$

Furthermore, we introduce an individualised valuation system that assigns individualised values $V(a,z) \in \mathbb{R}$ to any agent a for any collective good configuration $z \in \mathcal{Z}$. The total assigned value $\int V(\cdot, z)\, d\mu$ can be interpreted as the total resources available in the community for the provision of collective good configuration z.

An allocation consists of a collective good configuration $z \in \mathcal{Z}$, a provision plan $c \in C(z)$, consumption bundles f, and production plans g. Now, allocation (f, g, c, z) is *supported* by a valuation (p, V) if the following conditions hold:

Material balance: $\int f\, d\mu + c = \int g\, d\mu$

Budget balance: $\int V(\cdot, z)\, d\mu = p(z) \cdot c$;

Optimal provision: $\int V(\cdot, z)\, d\mu - p(z) \cdot c = \max_{z' \in \mathcal{Z}} \max_{c' \in C(z')} \int V(\cdot, z')\, d\mu - p(z') \cdot c'$, and;

Individual optimality: For every individual a it holds that $V(a,z) \leqslant p(z) \cdot g(a) = \sup p(z) \cdot \mathcal{P}_a(z)$ and any better arrangement is unaffordable for this agent in the sense that

$$(f', z') \succ_a (f(a), z) \text{ implies that } p(z)' \cdot f' + V(a, z') > \sup p(z') \cdot \mathcal{P}_a(z').$$

We claim that these four conditions describe the essence of collective good valuation in a wide range of applications in the literature.

2.1 The provision of a classic public service: Policing

To show the introduced conception, we apply this framework to an example of a traditional, Samuelsonian public service, namely that of community policing. Policing is a typical service provided through a professional class of government officers. This directly links the provision of this service to an appropriate configuration of a corresponding social division of labour.

Formally, the community itself is represented by the unit continuum $A = [0,1]$ endowed with the standard Euclidean topology and the Lebesgue measure μ. The social division of labour now takes the form of a (measurable) partitioning of A of specialised economic agents.

To simplify, we assume that there are two private goods: X (say, food) and Y, a form of human capital required in the provision of "policing" to the community A. Each agent $a \in A$ can choose between becoming a "farmer" and producing $(1,0)$ or a "police officer" and producing $(0,1)$. Hence, individual productive abilities are represented by the common production set $\mathcal{P} = \{(1,0), (0,1)\}$. Therefore, the resulting social division of labour based on assignment g is now given by $C_x = \{a \in A \mid g(a) = (1,0)\}$ and $C_y = A \setminus C_x = \{a \in A \mid g(a) = (0,1)\}$.

In this simple model, the level of policing is delivered by a professional class of police officers who are employed by a public authority. Hence, the public authority converts an input of human capital Y corresponding to policing services into a level of policing in the community. The total level of policing or "communal security" is given by the total fraction of the population that specialises as a police officer.

The choice to become a farmer or police officer is individualised and, therefore, effectuated within a (competitive) labour market. This implies that police officers are compensated with competitive wages and the government is obliged to deliver policing at a fair wage bill. A formal representation is that the government selects policing at level $0 \leqslant z \leqslant 1$ and that its provision costs are represented by a commodity bundle $c(z) = (0, z)$. The government has a wage bill of $p(z) \cdot c(z) = w_p\, z$, where $w_p \geqslant 0$ is the competitive wage of a police officer in the labour market.

To summarise, this framework is now represented by $\mathcal{Z} = [0, 1]$, a production correspondence given by $\mathcal{P}_a(z) = \mathcal{P} = \{(1, 0), (0, 1)\}$, and a collective cost function given by $c(z) = (0, z)$ for all $a \in A$ and $z \in \mathcal{Z}$.

This economy is completed by the introduction of preferences that are represented by a utility function $u\colon A \times \mathbb{R}^2_+ \times \mathcal{Z} \to \mathbb{R}$. For simplicity we let preferences be represented by a simple utility function with $u_a(x, y; z) = U(x, y; z) = xz$ for all $a \in A$, $(x, y) \in \mathbb{R}^2_+$ and $z \in \mathcal{Z}$.[6] We establish the following insight:

Proposition 2.1 *The allocation (f^*, g^*, z^*) given by*

$$f^*(a) = \left(\tfrac{1}{2}, 0\right), \qquad g^*(a) = \begin{cases} (1,0) & \text{if } a \leqslant \tfrac{1}{2}, \\ (0,1) & \text{if } a > \tfrac{1}{2}, \end{cases} \qquad z^* = \tfrac{1}{2} \text{ with cost } c(z^*) = \left(0, \tfrac{1}{2}\right).$$

is Pareto optimal. Moreover (f^*, g^*, z^*) can be supported by the price system given by $p(z) = (1, 1)$ for all $z \in \mathcal{Z}$ and a tax system given by $V(a, z) = z$ for all $a \in A$ and $z \in \mathcal{Z} = [0, 1]$.

For a proof of Proposition 2.1 we refer to Appendix A.1.

2.2 A governed commons

Hardin (1968) made the phrase *the tragedy of the commons* a "totemic reference to which tributes are regularly paid [. . .] in biology, ecology, and various social sciences" (Frischmann, Marciano, and Ramello, 2019). Hardin only considered two extreme solutions to the problem of overuse of the commons that arises from free-riding by economic agents: (i) regulation by the government and (ii) privatisation (Frischmann, Marciano, and Ramello, 2019, page 216).

Based on a multiyear interdisciplinary research project on this question, Ostrom (1990) challenged Hardin's reductionism and pointed out that some societies had spontaneously evolved local governance solutions to commons problems they had faced for centuries, solutions that involved behavioral norms and mutual monitoring. Some of these solutions worked well, others not.

[6] In this formalisation of the policing example, the collective good is Samuelsonian in nature due to the monotonicity of the utility functions in z.

To see how efficient governance of a commons—whether spontaneously communally evolved or centrally imposed—could be established, we consider a simple model of a community that is endowed with a productive commons. Usage rights to the commons are assumed to be non-tradable,[7] but critical in the production of the main consumable Y. We assume that an authority regulates the allocation of usage rights to the various individuals in the community. Our model is based on the following hypotheses and conceptions:

- As before, the set of agents is the unit interval $A = [0, 1]$ endowed with the standard Euclidean topology and the Lebesgue measure μ.
- There are two (tradable) commodities, a composite good X and a main consumable Y.
- The composite good X is produced by specialised consumer-producers.[8]
- The main consumable Y—interpreted to be an agricultural output—is produced through accessing the commons. Usage rights to the commons are allocated to the agents in the economy through a communal authority or government.
- The available commons has a total size of $0 < \Gamma < 1$, which is fully usable in the production of Y. The commons is assumed to be available costlessly.
- Every agent $a \in A$ is assigned a right to use the commons represented by a factor $0 \leqslant \gamma_a \leqslant 2\Gamma$.[9] The allocation of rights γ is bounded by the size of the available commons in the sense that $\int_0^1 \gamma_a \, da = \Gamma < 1$.

These assumptions translate to the following mathematical model:

- The set of all allocations of commons usage rights is given by

$$\mathcal{Z} = \left\{ \gamma : A \to [0, 2\Gamma\,] \;\middle|\; \gamma \text{ is integrable and } \int_0^1 \gamma_a \, da \leqslant \Gamma \right\}$$

  such that $c(\gamma) = (0, 0)$ for every $\gamma \in \mathcal{Z}$;

- Every agent $a \in A$ has productive abilities represented by the production set $\mathcal{P}_a(\gamma) = \{(1, 0), (0, \gamma_a)\}$ and preferences given by $u_a(x, y, z) = xy$.

The following proposition introduces a supported optimal allocation in this community. For a proof of Proposition 2.2 we refer to Appendix A.2.

[7] In principle, this refers to a situation that can be improved upon significantly by the introduction of tradable usage rights as argued in the seminal contribution by Coase (1960).

[8] The composite good X could be thought of as a consumable proxy for shelter and other amenities that are produced through skilled labour. Therefore, X-producers can be considered as skilled workers, represented by their trade guilds. Examples of these skilled workers are carpenters, builders and blacksmiths. Their outputs—furniture, dwellings and tools—are captured in the abstract category X.

[9] Individual usage rights are assumed to be bounded by 2Γ. This bound is only introduced to allow for manageable computations and derivations. It essentially excludes the allocation of arbitrarily large individual usage rights.

Proposition 2.2 *Let (f^*, g^*, γ^*) be as follows*

$$\begin{cases} f^*(a) = \left(\frac{3}{4}, \frac{3\Gamma}{2}\right) & g^*(a) = (1,0) & \gamma^*(a) = 0 & \text{for } a \leqslant \frac{1}{2} \\ f^*(a) = \left(\frac{1}{4}, \frac{\Gamma}{2}\right) & g^*(a) = (0, \gamma^*(a)) & \gamma^*(a) = 2\Gamma & \text{for } a > \frac{1}{2} \end{cases}$$

Then (f^*, g^*, γ^*) is Pareto optimal and can be supported by (p, V) with for every $\gamma \in \mathcal{Z}$:

$$p(\gamma) = (2\Gamma, 1) \text{ for any } \gamma \in \mathcal{Z} \quad \text{and} \quad V(a,\gamma) = \begin{cases} -\Gamma & \text{for } a \leqslant \frac{1}{2} \\ \Gamma & \text{for } a > \frac{1}{2} \end{cases}$$

The proposition identifies a supported situation in which half of the population provides skilled human capital to produce X, while the other half equally use the communal commons to produce Y. In this equilibrium, there is a usage tax imposed on all Y-producers, which is transferred as compensation to the X-producers in the social division of labour. V can interpreted as an implicit valuation of the commons, even though the usage of the commons is not tradable, but managed outside of markets.

2.3 A knowledge economy

The role of public education on economic wealth creation in a system of social division of labour has been recognised throughout the literature. Buchanan and Yoon (2002) point out that the Smithian logic of the social division of labour explicitly recognises the role of knowledge and knowledge sharing at the foundation of productive abilities in the economy.

Hayek (1937) and Becker and Murphy (1992) have addressed the role of knowledge in the social division of labor, but focus on how knowledge is shared among specialised entities that make up the social division of labor. Hayek's theory is well-known and discusses how markets act as optimal information processors, thereby implicitly the role of market systems as collective goods. This refers to transaction cost arguments related to market functionality such as considered in Diamantaras, Gilles, and Ruys (2003) and Gilles and Diamantaras (2003).

Becker and Murphy (1992, page 1140) employ a model of the economy with a continuum of tasks that can be used by teams of intrinsically identical workers to produce a single consumption good. Here, workers differentiate themselves endogenously by specializing through team membership. Becker and Murphy consider productivity gains from specialization as well as conflicts that make the coordination of larger teams of workers more costly. They do so by positing a production and a coordination cost function that is increasing in the size of a production team. They discuss optimal allocations under the assumption that market forces will support them, since they rule out externalities.

Our analysis instead focuses on exactly what equilibrium concept decentralizes optimal allocations in a model with multiple private goods and heterogeneous consumer-producers, and with widespread externalities, which gives a much more fine-grained insight on the problem of knowledge as a widespread externality, as can be more fully appreciated by examining the general model we lay out after this example. However, we do not discuss coordination costs, as we have left out of our framework transaction costs. We also do not discuss dynamic issues, as our model is static.

We focus on the collective good nature of knowledge. Therefore, our model links to contemporary issues on the role of education in the economy. Refinements of the simple model considered here would also allow us to study the implications of education policy for the evolution of the social division of labour.

Knowledge as a collective good. We assume that knowledge is freely accessible in the economy, but that it is costly to create knowledge. Thus, knowledge is provided through government intervention such as considered by Mazzucato (2014). The provided knowledge impacts productivity, but has no direct externalities on the preferences of the agents.

As before, let the set of agents be represented by $A = [0, 1]$ endowed with the Euclidean topology and the Lebesgue measure. There are two consumable commodities, food X and a composite good Y. The production of Y is subject to the overall level of knowledge in the economy. The level of knowledge is denoted by $z \in \mathcal{Z} = [0, 1]$. It is costly to generate knowledge, represented by a cost correspondence with

$$C(z) = \left\{ \left(\frac{z+\lambda}{z+1}\, z \, , \, \frac{(1-\lambda)z}{z+1}\, z \right) \middle| \; 0 \leqslant \lambda \leqslant 1 \right\}$$

In this model, the same level of productive knowledge can be created by the use of different combinations of resources, from the equal use of both goods—represented by $\lambda = 0$—to the sole use of food—represented by $\lambda = 1$. The choice of λ to deliver the collective knowledge level z is now part of the allocation itself.

Every $a \in A$ has productive abilities described by the production set $\mathcal{P}_a(z) = \{(1, 0), (0, z)\}$, implying that an agent can be a farmer—producing food X only—or a skilled worker using knowledge to produce the composite durable good Y. Furthermore, all agents desire food and durable goods in equal measure, represented through the common Cobb-Douglas utility function $u_a(x, y, z) = xy$.

An allocation is now typically described as a tuple (f, g, λ, z) that comprises of a collective level of productive knowledge $z \in \mathcal{Z}$, a consumptive allocation of goods $f : A \to \mathbb{R}^2_+$, a production plan g assigning $g(a) \in \mathcal{P}_a(z)$ and a provision plan for the selected collective level of knowledge z represented by $0 \leqslant \lambda \leqslant 1$.

Equilibrium pricing. Consider an allocation $(f^*, g^*, \lambda^*, z^*)$ that is supported by some price-valuation system (p, V) with $p = (p_x, p_y)$. Due to the nature of the utility function, it is easy to see that both available professions have to be viable. Thus, both professions should generate an equal income: $p_x(z^*)\,1 = p_y(z^*)z^*$. Therefore, we deduce that $p_x(z^*) = z^*$, $p_y(z^*) = 1$ and $p(z^*) \cdot g = z^*$ for every $g \in \mathcal{P}_a(z^*)$ and every $a \in A$. Thus, for every $z \in \mathcal{Z}$: $p(z) = (z, 1)$ implying $\max p(z) \cdot \mathcal{P}_a(z) = z$ and for every $a \in A$: $p(z^*) \cdot g^*(a) = \max p(z^*) \cdot \mathcal{P}_a(z^*) = z^*$ and $(f^*(a), z^*)$ maximises u_a on the budget set

$$\{ (f, z) \mid p(z) \cdot f + V(a, z) \leqslant \max p(z) \cdot \mathcal{P}_a(z) \} = \{ (x, y, z) \mid zx + y + V(a, z) \leqslant z \} .$$

In particular, for $z = z^*$ we derive that the optimal consumption bundles for the formulated budget sets are given by

$$f^*(a) = \left(\frac{z^* - V(a, z^*)}{2z^*}, \frac{z^* - V(a, z^*)}{2} \right)$$

Furthermore, note that $p(z) \cdot c = z^2$ for every provision plan $c \in C(z)$. Therefore, public budget balance described by $\int V(a, z^*)\, da = p(z^*) \cdot c^* = (z^*)^2$ now implies that

$$\int f^*(a)\, da = \left(\frac{z^* - \int V(a, z^*)\, da}{2z^*}, \frac{z^* - \int V(a, z^*)\, da}{2} \right) = \left(\frac{1 - z^*}{2}, \frac{z^*(1 - z^*)}{2} \right).$$

Finally, there emerges a social division of labor described by $\mu_x = \mu\left(\{a \in A \mid g^*(a) = (1, 0)\}\right)$ and $\mu_y = \mu\left(\{a \in A \mid g^*(a) = (0, z^*)\}\right)$ where μ is the Lebesgue measure on $A = [0, 1]$.

This computation results in the following proposition that identifies a continuum of supported socially optimal allocations in this knowledge economy:

Proposition 2.3 *Let $0 \leqslant \lambda \leqslant 1$ and define the allocation $A_\lambda = (f^*, g^*, \lambda, z^*)$ by $z^* = \frac{1}{3}$, $f^*(a) = \left(\frac{1}{3}, \frac{1}{9}\right)$ and*

$$g^*(a) = \begin{cases} (1, 0) & \text{for } a \leqslant \frac{5}{12} + \frac{1}{4}\lambda \\ \left(0, \frac{1}{3}\right) & \text{for } a > \frac{5}{12} + \frac{1}{4}\lambda \end{cases}$$

Then the family $\{A_\lambda \mid 0 \leqslant \lambda \leqslant 1\}$ is supported by a price-valuation system (p, V) with $p(z) = (z, 1)$ and $V(a, z) = z^2$ for every $a \in A$ and $z \in [0, 1]$ through a social division of labor given by $\mu_x = \frac{5}{12} + \frac{1}{4}\lambda$ and $\mu_y = \frac{7}{12} - \frac{1}{4}\lambda$.

For a proof of Proposition 2.3 we refer to Appendix A.3.

In this knowledge economy, there exists a continuum of supported Pareto optimal allocations, depending on the selected provision plan for the required level of knowledge $z^* = \frac{1}{3}$. We derived these equilibria through "reverse engineering": Income equalisation determines equilibrium prices for each $z \in \mathcal{Z}$; this, in turn, determines equilibrium demand for $z \in \mathcal{Z}$; and, finally, the equalisation of demand and supply at these prices results in a certain equilibrium social division of labour. The optimisation of the utility values for all agents over $z \in \mathcal{Z}$ then identifies an equilibrium.

3 A general theory

We consider an economy with a diversified production sector based on the hypothesis that all agents are participating directly in the production as well as the consumption of goods.

Private goods. Formally, we consider $\ell \geqslant 1$ tradable private commodities.[10] Hence, the *private commodity space* is represented by the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$. The commodity space represents all bundles of tradable or "marketable" goods in this economy.

[10] In particular, if $\ell = 1$ we have a framework akin to the one developed in Mas-Colell (1980).

For $k = 1, \ldots, \ell$ we denote by $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ the k-th unit bundle in $\mathbb{R}^\ell_+$ and by $e = (1, \ldots, 1)$ the bundle consisting of one unit of each commodity.[11]

We emphasise that these ℓ commodities particularly include diversified forms of human capital, in the form of professionally trained specialists (Yang, 2001). These forms of specialised human capital can be employed to represent the delivery of public services in the economy.

Collective goods. We assume that a *public authority* provides collective goods to the community of consumer-producers using the tradable resources generated by that community. The authority pays market prices for these inputs.

Formally, we let $z \in \mathcal{Z}$ represent a configuration of collective goods provided by the public authority, where $\mathcal{Z}$ is some abstract provision space as considered seminally in Mas-Colell (1980) and Diamantaras and Gilles (1996). Consequentially, the collective good configurations introduced here generalise Samuelson's quantifiable notion of a public good (Samuelson, 1954).

Following the framework set out in Diamantaras, Gilles, and Scotchmer (1996), we assume that collective good configurations can be delivered by the public authority through a variety of provision plans. Hence, the input requirement for the provision of collective good configurations is modelled as a cost correspondence $C \colon \mathcal{Z} \twoheadrightarrow \mathbb{R}^\ell_+$, where $C(z) \neq \varnothing$ is the set of all possible input commodity bundles required for the provision of collective good configuration $z \in \mathcal{Z}$.[12] In particular, different input commodity bundles—composed of different mixtures of commodity inputs—can be selected and implemented to deliver the same collective good configuration. A typical input vector for collective good configuration $z \in \mathcal{Z}$ is denoted as $c \in C(z)$.

3.1 Introducing consumer-producers

The set of economic agents is denoted by A and a typical economic agent is denoted by $a \in A$. Throughout, we let $\Sigma \subset 2^A$ be a σ-algebra of measurable coalitions in A and we let the function $\mu \colon \Sigma \longrightarrow [0, 1]$ be a complete probability measure on (A, Σ). We use a very general setup based on the path-breaking theory of large economies in Hildenbrand (1969).

The next definition formalises the notion of a consumer-producer.[13]

Definition 3.1 *Every agent $a \in A$ is modelled as a* ***consumer-producer****, endowed with consumptive* as well as *productive abilities, represented as triple* $(X_a, \mathcal{P}_a, \succsim_a)$ *where*

- $X_a \colon \mathcal{Z} \twoheadrightarrow \mathbb{R}^\ell_+$ *is a's consumption set correspondence that assigns to every configuration of the collective good $z \in \mathcal{Z}$ a consumption set $X_a(z) \subset \mathbb{R}^\ell_+$ consisting of private good bundles that are accessible to agent a;*

[11] Throughout, we employ the vector inequality notation in which $x \geqslant x'$ if $x_k \geqslant x'_k$ for all commodities $k = 1, \ldots, \ell$; $x > x'$ if $x \geqslant x'$ and $x \neq x'$; and $x \gg x'$ if $x_k > x'_k$ for all commodities $k = 1, \ldots, \ell$.

[12] In this formulation we use the notion of a point-to-set correspondence. Formally, a point-to-set correspondence from some abstract set X to the ℓ-dimensional Euclidean space is represented as $\mathcal{F} \colon X \twoheadrightarrow \mathbb{R}^\ell$ which can be denoted alternatively by $\mathcal{F} \colon X \longrightarrow 2^{\mathbb{R}^\ell}$. Here, the correspondence $\mathcal{F}$ assigns to every $x \in X$ some set $\mathcal{F}(x) \subset \mathbb{R}^\ell$.

[13] For a more detailed development and discussion of the concept of a consumer-producer we refer to Yang (2001) and Gilles (2019b).

- $\mathcal{P}_a \colon \mathcal{Z} \twoheadrightarrow \mathbb{R}^\ell$ *is a's production correspondence that assigns to every configuration of the collective good* $z \in \mathcal{Z}$ *a production set* $\mathcal{P}_a(z) \subset \mathbb{R}^\ell$ *consisting of input-output bundles that agent a can produce;*
- *and* $\succsim_a \subset \left(\mathbb{R}^\ell_+ \times \mathcal{Z}\right) \times \left(\mathbb{R}^\ell_+ \times \mathcal{Z}\right)$ *is a reflexive binary relation representing a's consumptive preferences.*

We discuss next in some detail the model of an economic agent as a consumer-producer. The consumptive factors related to an economic agent $a \in A$ are represented by the pair $(X_a, \succsim_a)$, while her productive abilities are represented by $\mathcal{P}_a$.

Consumption. The consumption set correspondence X assigns to every agent $a \in A$ and every collective good configuration $z \in \mathcal{Z}$ a set of available or accessible private good bundles $X_a(z) \subset \mathbb{R}^\ell_+$. We model this restriction of the consumption set completely *independently* of the agent's consumptive preferences, which are defined (as far as the private goods are involved) on the space of *all* nonnegative potential private good bundles $\mathbb{R}^\ell_+$. That is, the agents can envision any nonnegative consumption vector of private goods, alongside every collective good $z \in Z$, when comparing consumption bundles, even though some consumption bundles might involve unavailable private good vectors.

For any consumer-producer $a \in A$, the binary relation $\succsim_a$ has the standard interpretation: $(x, z) \succsim_a (x', z')$ means that the consumptive configuration $(x, z) \in \mathbb{R}^\ell_+ \times \mathcal{Z}$ is *at least as good as* consumptive configuration $(x', z') \in \mathbb{R}^\ell_+ \times \mathcal{Z}$. We emphasise that we do not impose any conditions on these preferences such as completeness, transitivity or continuity.

We denote by

$$\left\{(x', z') \in \mathbb{R}^\ell_+ \times \mathcal{Z} \mid (x', z') \succsim_a (x, z)\right\} \tag{1}$$

the *weak better set of* (x, z) consisting of all consumption configurations that a assesses as at least as good as (x, z).

Let $(x, z) \succ_a (x', z')$ if $(x, z) \succsim_a (x', z')$ and *not* $(x', z') \succsim_a (x, z)$. This is interpreted that (x, z) is assessed as strictly better than (x', z') by consumer-producer $a \in A$. We introduce the *(strict) better set* of (x, z) for a as

$$\left\{(x', z') \in \mathbb{R}^\ell_+ \times \mathcal{Z} \mid (x', z') \succ_a (x, z)\right\} \subset \left\{(x', z') \in \mathbb{R}^\ell_+ \times \mathcal{Z} \mid (x', z') \succsim_a (x, z)\right\}. \tag{2}$$

The following definition formalises the non-satiation property of consumptive preferences in our context.

Definition 3.2 *Let* $(X_a, \mathcal{P}_a, \succsim_a)$ *represent some agent* $a \in A$ *as a consumer-producer.*
We say that agent $a \in A$ *is* ***non-satiated at*** (x, z) ***regarding*** z' *if there exists some* $x' \in X_a(z')$ *such that* $(x', z') \succ_a (x, z)$.
We say that agent $a \in A$ *is* ***non-satiated at*** (x, z) *if a is non-satiated at* (x, z) *regarding z itself.*

For $a \in A$, we say that a utility function $u_a \colon \mathbb{R}^\ell_+ \times \mathcal{Z} \to \mathbb{R}$ *represents* the preference relation $\succsim_a$ whenever $(x, z) \succsim_a (x', z')$ if and only if $u_a(x, z) \geqslant u_a(x', z')$. Clearly, if a preference is represented by a utility function, it is complete and transitive.

Production. Throughout we assume that the selection of the collective good configuration directly affects individual productive abilities. Hence, the production set assigned to an individual consumer-producer is affected by $z \in \mathcal{Z}$ and represented as a correspondence $\mathcal{P}_a : \mathcal{Z} \twoheadrightarrow \mathbb{R}^\ell$. Although collective goods are not used as direct inputs in any production process, collective goods impose widespread externalities on productive abilities, as reflected in the assumed dependence of individual production sets on z.

For every $a \in A$ and $z \in \mathcal{Z}$, we assume that a typical production bundle $y \in \mathcal{P}_a(z)$ can be written as $y = y^+ - y^-$ where $y^+ = y \vee 0$ denotes the outputs of a's production process and $y^- = (-y) \vee 0$ denotes the tradable inputs required for producing y^+.

This formulation of production has been developed in Gilles (2019b) and extends the standard approach in economies with consumer-producers developed in Yang (2001); Sun, Yang, and Zhou (2004) and Diamantaras and Gilles (2004), in which all production is achieved through the use of non-tradable inputs only. This approach can be recovered by imposing that $y^- = 0$, letting $y = y^+ > 0$ be a vector of outputs only, i.e., the production of tradable outputs is based on the usage of non-tradable, privately owned inputs only.

The next definition brings together the regularity properties that one may expect to be satisfied by a production set.

Definition 3.3 *Consider a production set $\mathcal{P} \subset \mathbb{R}^\ell$. We introduce the following terminology:*

(i) *The production set $\mathcal{P}$ is **regular** if $\mathcal{P}$ is a closed set, $\mathcal{P}$ is bounded from above, $0 \in \mathcal{P}$ and $\mathcal{P}$ is comprehensive in the sense that*

$$\mathcal{P} - \mathbb{R}^\ell_+ \equiv \left\{ y - y' \, \middle| \, y \in \mathcal{P} \text{ and } y' \in \mathbb{R}^\ell_+ \right\} \subset \mathcal{P}. \tag{3}$$

(ii) *The production set $\mathcal{P}$ is **delimited** if there exists a compact set $\overline{\mathcal{P}} \subset \mathbb{R}^\ell$ such that $0 \in \overline{\mathcal{P}}$ and*

$$\mathcal{P} = \overline{\mathcal{P}} - \mathbb{R}^\ell_+. \tag{4}$$

A regular production set satisfies two basic properties used throughout general equilibrium theory, namely the ability to cease production altogether and the assumption of free disposal in production. Furthermore, it is natural to assume that an individual economic agent can only generate a bounded total output.

Note that we do *not* impose convexity of the production sets. Indeed, regular production sets could be convex and satisfy decreasing returns to scale, but, on the other hand, regular production sets also might exhibit increasing returns to scale—subject to boundedness of the generated output—and increasing returns to specialisation, allowing the kinds of production sets developed in the literature on market economies with an endogenous social division of labour (Yang, 1988; Gilles, 2019b).

Delimitedness of a production set is defined by applying the free-disposal property to a compact set of core production points $\overline{\mathcal{P}} \subset \mathbb{R}^\ell$. Delimited production sets are obviously regular.

3.2 Collective good economies with consumer-producers

We conclude our discussion of the various elements of our model by introducing a comprehensive descriptor of an economy in which the provision of collective goods is facilitated through a social division of labour founded on consumer-producers.

Definition 3.4 *An **economy** is a list* $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, X, \succsim, \mathcal{P}, C \rangle$ *where*

- (A, Σ, μ) *is a complete atomless probability space of economic agents;*
- $\mathcal{Z}$ *is a set of collective good configurations, a set that need not be endowed with a topology or an order;*
- $X \colon A \times Z \twoheadrightarrow \mathbb{R}^\ell_+$ *assigns to every agent* $a \in A$ *and collective good configuration* $z \in \mathcal{Z}$ *a non-empty consumption set* $X_a(z) \neq \varnothing$*;*
- $\succsim_a \subset \left(\mathbb{R}^\ell_+ \times \mathcal{Z}\right) \times \left(\mathbb{R}^\ell_+ \times \mathcal{Z}\right)$ *is a preference relation for every* $a \in A$ *such that for every collective good configuration* $z \in \mathcal{Z}$ *and every integrable assignment of private goods* $f \colon A \to \mathbb{R}^\ell_+$ *with* $f(a) \in X_a(z)$ *it holds that for every* $z' \in \mathcal{Z}$*:*

$$\left\{(a, x') \in A \times \mathbb{R}^\ell_+ \mid x' \in X_a(z')\,,\ (x', z') \succ_a (f(a), z)\right\} \in \Sigma \otimes \mathcal{B}\left(\mathbb{R}^\ell\right) \quad \textit{and} \tag{5}$$

$$\left\{(a, x') \in A \times \mathbb{R}^\ell_+ \mid x' \in X_a(z')\,,\ (x', z') \succsim_a (f(a), z)\right\} \in \Sigma \otimes \mathcal{B}\left(\mathbb{R}^\ell\right), \tag{6}$$

  imposing two natural measurability conditions on the preferences; [14]

- $\mathcal{P} \colon A \times \mathcal{Z} \twoheadrightarrow \mathbb{R}^\ell$ *is a correspondence that assigns to every agent* $a \in A$ *and collective good configuration* $z \in \mathcal{Z}$ *a regular production set* $\mathcal{P}_a(z) := \mathcal{P}(a, z) \subset \mathbb{R}^\ell$ *such that for every* $z \in \mathcal{Z}$*,* $\mathcal{P}(\cdot, z) \colon A \twoheadrightarrow \mathbb{R}^\ell$ *has a measurable graph in* $\Sigma \otimes \mathcal{B}(\mathbb{R}^\ell)$*;*
- *and* $C \colon \mathcal{Z} \twoheadrightarrow \mathbb{R}^\ell_+$ *is a provision cost correspondence assigning to each collective good configuration a set of possible input vectors such that for every collective good configuration* $z \in \mathcal{Z}$*:*
  - (i) $C(z) \neq \varnothing$ *is a convex set and;*
  - (ii) *there exist some input vector* $c \in C(z)$ *and some integrable function* $g \colon A \to \mathbb{R}^\ell$ *such that* $g(a) \in \mathcal{P}_a(z)$ *for all* $a \in A$ *and* $c \leqslant \int g \, d\mu$*.*

This definition explicitly assumes that the provision of any configuration of collective goods directly affects the consumptive preferences and the production set of every agent in the economy, as already noted. In this way, we allow for widespread externalities that emanate from the collective good provision throughout the economy. This opens the door for using our model to study the impact of education and regulation on economic performance.

Furthermore, the collective good configurations considered are only those that can be provided for through the production of the required inputs.

[14]Here, $\mathcal{B}(\mathbb{R}^\ell)$ denotes the σ-algebra of Borel sets generated by the Euclidean topology of open sets in $\mathbb{R}^\ell$ and, consequently, $\Sigma \otimes \mathcal{B}(\mathbb{R}^\ell)$ denotes the σ-algebra of measurable sets generated by all measurable sets on A and the Borel sets in $\mathbb{R}^\ell$. We remark that, if $\succsim_a$ is represented by $u(a, \cdot, z)$ for every $a \in A$ and $z \in \mathcal{Z}$, this condition converts to the standard joint measurability condition of that function on (A, Σ, μ) and the Euclidean space $\mathbb{R}^\ell$.

Remark 3.5 Our definition of an economy is quite general and covers some well-established existing frameworks in the literature as special cases.

Private good endowment economies with collective goods: Our framework includes collective good economies founded on an initial endowment of private commodities. Formally, consider an initial endowment of private goods as an integrable function $w\colon A \to \mathbb{R}^{\ell}_{+}$ with $\int w\, d\mu \gg 0$. Now, let $\widetilde{\mathcal{P}}\colon A \times \mathcal{Z} \twoheadrightarrow \mathbb{R}^{\ell}$ be a measurable production correspondence such that $\widetilde{\mathcal{P}}_a(z)$ is compact and $\widetilde{\mathcal{P}}_a(z) \cap \mathbb{R}^{\ell}_{+} = \varnothing$. Then $\widetilde{\mathcal{P}}_a(z)$ represents a pure transformation production technology that transforms certain input quantities of private goods into certain private good output quantities.

Now, for every $a \in A$, the production set is defined as $\mathcal{P}_a(z) = \left(\{w(a)\} + \widetilde{\mathcal{P}}_a(z) \right) \cup \{0\} - \mathbb{R}^{\ell}_{+}$. Clearly, this construction converts the private good endowment economy into that of a delimited production correspondence as part of an economy in the sense of Definition 3.4. Indeed, we note that $0 \in \mathcal{P}_a(z)$. Therefore, this construction covers the case of a collective good economy with an initial endowment of private goods in which economic agents have access to a production technology to convert these endowments into a range of outputs.

Also, if the production technology descriptors introduced above are given by $\widetilde{\mathcal{P}}_a(z) = \varnothing$ and $w(a) \gg 0$ for all $a \in A$ and $z \in \mathcal{Z}$, the economy reverts to that of an economy with an initial endowment only, that is, an economy as considered in the literature on standard exchange economies with collective goods (Mas-Colell, 1980; Diamantaras and Gilles, 1996; Diamantaras, Gilles, and Scotchmer, 1996; Gilles and Diamantaras, 1998).

Home production economies: Our framework also captures the case that production can be based on the allocation of one unit of non-marketable labour time over ℓ different (home) production processes, each generating the output of a marketable commodity such as considered in Sun, Yang, and Zhou (2004), Cheng and Yang (2004) and Diamantaras and Gilles (2004) .

Formally, for every collective good configuration $z \in \mathcal{Z}$ and every private commodity $k \in \{1, \ldots, \ell\}$ we let $f^z_k\colon [0,1] \to \mathbb{R}_{+}$ be a production function converting a quantity of invested labour time in the output of the k-th commodity. We impose that f^z_k is continuous, that $f^z_k(L) > 0$ for all positive labour inputs $L > 0$ and that $f^z_k(0) = 0$. Now let $F^z = (f^z_1, \ldots, f^z_\ell)\colon [0,1]^\ell \to \mathbb{R}^{\ell}_{+}$. Now define a production set as follows

$$\mathcal{P}(F^z) = \overline{\mathcal{P}}(F^z) - \mathbb{R}^{\ell}_{+} \subset \mathbb{R}^{\ell},$$

where

$$\overline{\mathcal{P}}(F^z) = \left\{ F^z(L_1, \ldots, L_\ell) \;\middle|\; \sum_{k=1}^{\ell} L_k \leqslant 1 \right\} \subset \mathbb{R}^{\ell}_{+}.$$

Note that by definition $\mathcal{P}(F^z)$ is a home-based production set.[15] Therefore, if production is based on the allocation of labour time over production processes for all ℓ commodities, we arrive at a situation that is captured by our concept of home-based production sets.

[15] We refer to Gilles (2019b) for a formal definition of the notion of home-based production.

Private ownership production economies: Our formulation of an economy also covers the so-called private ownership production economies with collective goods (De Simone and Graziano, 2004; Graziano, 2007). For simplicity, we consider a private ownership production economy with ℓ private goods. As before, an abstract set $\mathcal{Z}$ represents all collective good configurations. A cost correspondence $C \colon \mathcal{Z} \to 2^{\mathbb{R}^\ell_+}$ represents all plans for the realisation of a collective good configuration in terms of required private good inputs.

There is a finite number of consumers, i.e., $A = \{1, \dots, M\}$, each of whom is characterised by consumptive preferences $\succsim_a \subset \left(\mathbb{R}^\ell_+ \times \mathcal{Z}\right) \times \left(\mathbb{R}^\ell_+ \times \mathcal{Z}\right)$ and an initial endowment of private goods $\omega_a \in \mathbb{R}^\ell_+$. The total private good endowment in the economy is expressed by $\omega = \sum_{a \in A} \omega_a$.

There is a finite set $J = \{1, \dots, K\}$ of producers. Any producer $j \in J$ is represented by a compact production set $Y_j \subset \mathbb{R}^\ell$. The profits of each producer j are shared among consumers according to a share function. In particular, the shares of consumer $a \in A$ in the profit of $j \in J$ are denoted by $\theta_{aj} \in [0, 1]$ with $\sum_{a \in A} \theta_{aj} = 1$.

We now represent this economy as an economy $\mathbb{E}$ using our framework. Let $A = \{1, \dots, M\}$ be the set of *consumer-producers*. Every individual $a \in A$ is endowed with $(\succsim_a, X_a, \mathcal{P}_a)$, where $\succsim_a$ is the preference relation as introduced above, $X_a(z) = \mathbb{R}^\ell_+$ assigns the standard consumption space to agent a for all $z \in \mathcal{Z}$, and $\mathcal{P}_a \colon \mathcal{Z} \twoheadrightarrow \mathbb{R}^\ell$ is a's production correspondence assigning to every collective good configuration $z \in \mathcal{Z}$ the following production set:

$$\mathcal{P}_a(z) = \left(\sum_{j \in J} \theta_{aj} Y_j + \{\omega_a\} \right) \cup \{0\} - \mathbb{R}^\ell_+ .$$

We remark that $\mathcal{P}_a(z)$ is regular. Thus, the constructed economy $\mathbb{E}$ indeed represents the case of a private ownership production economy.

The notion of an economy introduced in Definition 3.4 covers the study of the emergence of a social division of labour in a production economy without collective goods. Indeed, if there is a trivial collective good structure configured as $\mathcal{Z} = \{z\}$ with $C(z) = \{0\}$, then the definition reverts to a continuum economy with an endogenous social division of labour considered in Gilles (2019b).

Finally, we mention that our model extends the theory put forward by Hildenbrand (1969). His model is a special case of ours in the sense that it can be viewed as a private ownership production economy with a trivial collective good structure. □

3.3 Allocations and Pareto optimality

We are now in a position to introduce the notion of an allocation and its feasibility in the context of an economy with collective goods $\mathbb{E}$ set out above.

Definition 3.6 *An **allocation** in the economy $\mathbb{E} = \langle (A, \Sigma, \mu), \mathcal{Z}, X, \succsim, \mathcal{P}, C \rangle$ is a tuple (f, g, c, z) where*

- *$z \in \mathcal{Z}$ is some collective good configuration that is provided in the economy;*
- *$c \in C(z)$ is some input vector for the generation of the selected collective good configuration z;*

- *$g\colon A \longrightarrow \mathbb{R}^\ell$ is an integrable function that assigns to every agent $a \in A$ a production plan $g(a) \in \mathcal{P}_a(z)$;*
- *and $f\colon A \longrightarrow \mathbb{R}^\ell_+$ is an integrable function that assigns to every agent $a \in A$ a private good consumption bundle $f(a) \in X_a(z)$.*

*An allocation (f, g, c, z) is **feasible** in the economy $\mathbb{E}$ if it holds that*

$$\int f \, d\mu + c = \int g \, d\mu. \tag{7}$$

Feasibility means that all private good consumption bundles as well as the cost of the provided collective good configuration can be covered by using the productive facilities and abilities that are present in the economy. We do not explicitly impose free disposal of private goods in the definition of feasibility; any disposal that may occur would happen via the comprehensiveness assumption on the production sets.

We conclude the introduction of the fundamental notions in our model with the standard notion of Pareto efficiency in this context.

Definition 3.7 *A feasible allocation (f, g, c, z) is **Pareto optimal** in the economy $\mathbb{E}$ if there is no feasible allocation (f', g', c', z') such that for almost every $a \in A$ it holds that $(f(a), z) \precsim_a (f'(a), z')$ and there exists a coalition $S \in \Sigma$ with $\mu(S) > 0$ such that $(f(b), z) \prec_b (f'(b), z')$ for all $b \in S$.*

The discussion in the next section focusses on the identification of conditions on an economy $\mathbb{E}$ that allow the support of Pareto optimal allocations through an appropriate system of pricing private goods and taxing the collective good consumption.

4 Valuation equilibrium

We extend the notion of valuation equilibrium (Mas-Colell, 1980) to our class of economies in which collective goods are provided through a social division of labour. In our valuation equilibrium concept, a feasible allocation is supported through a conditional private good price system as well as a "valuation system", representing a tax-subsidy scheme. We introduce this notion of equilibrium in two stages. First, we consider supporting price-valuation systems and, subsequently, we strengthen the definition to describe a full valuation equilibrium.

Definition 4.1 *A feasible allocation (f^*, g^*, c^*, z^*) in the economy $\mathbb{E}$ is **supported** by a price system $p\colon \mathcal{Z} \to \mathbb{R}^\ell_+ \setminus \{0\}$ and a valuation system $V\colon A \times \mathcal{Z} \to \mathbb{R}$ if:*

(i) *For every collective good configuration $z \in \mathcal{Z}$: $V(\cdot, z)$ is integrable and for almost every agent $a \in A$ it holds that $V(a, z) \leqslant \sup p(z) \cdot \mathcal{P}_a(z)$;*

(ii) *There is budget neutrality in equilibrium, i.e.,*

$$\int V(\cdot, z^*) \, d\mu = p(z^*) \cdot c^* \,;$$

(iii) *The pair* (c^*, z^*) *maximises the collective surplus in the sense that*

$$\int V(\cdot, z^*)\, d\mu - p(z^*) \cdot c^* = \max_{z \in \mathcal{Z}} \max_{c \in C(z)} \int V(\cdot, z)\, d\mu - p(z) \cdot c\,;$$

(iv) *And for almost every agent* $a \in A$ *if* $f \in X_a(z)$ *with* $(f, z) \succsim_a (f^*(a), z^*)$*, then*

$$p(z) \cdot f + V(a, z) \geqslant \sup p(z) \cdot \mathcal{P}_a(z).$$

The definition of a supporting price and valuation system imposes four support conditions.

A valuation system V imposes a tax on agent $a \in A$ if $V(a, z) > 0$ and it transfers a subsidy to $a \in A$ if $V(a, z) < 0$. Condition (i) now requires that all taxes ($V(a, z) > 0$) are in principle payable from income acquired from an appropriately selected production plan. This excludes that the public authority can impose infinitely large taxes on individuals to obstruct certain collective good configurations. There is no bound on the assignment of a subsidy to an agent.

Condition (ii) is a simple budget neutrality condition and condition (iii) imposes that the selected collective good configuration and the corresponding input vector is the surplus maximising configuration. Hence, in a supported allocation, the selected collective good configuration and the corresponding input vector results in budget balance, while out-of-equilibrium configurations would result in budgetary deficits.

Condition (iv) is a standard quasi-equilibrium condition that the expenditure of acquiring a better private good consumption bundle under an alternative collective good configuration either exceeds or is equal to the maximal income that an agent can generate by using her productive abilities under the prevailing private good prices.

Condition (iv) in the definition above can be replaced by a full equilibrium preference maximisation over any agent's budget set. This forms the basis for the next definition, that of a full valuation equilibrium.

Definition 4.2 *A feasible allocation* (f^*, g^*, c^*, z^*) *is a* ***valuation equilibrium*** *in the economy* $\mathbb{E}$ *if there exist a price system* $p\colon \mathcal{Z} \to \mathbb{R}^\ell_+ \setminus \{0\}$ *and a valuation system* $V\colon A \times \mathcal{Z} \to \mathbb{R}$ *such that properties* (i)–(iii) *of Definition 4.1 hold and additionally the following property replacing 4.1*(iv) *holds:*

(iv') and for almost every agent $a \in A$ *the tuple* $(f^*(a), g^*(a), z^*)$ *is a* $\succsim_a$*-optimal point in the budget set given by*

$$B(a, p, V) = \left\{ (f, g, z) \in \mathbb{R}^\ell_+ \times \mathbb{R}^\ell \times \mathcal{Z} \;\middle|\; \begin{array}{c} f \in X_a(z) \text{ and } g \in \mathcal{P}_a(z) \\ p(z) \cdot f + V(a, z) \leqslant p(z) \cdot g \end{array} \right\} \tag{8}$$

The valuation equilibrium concept is the natural equivalent of the standard competitive equilibrium concept in a continuum exchange economy extended to the context of an economy with an endogenous social division of labour and non-Samuelsonian collective goods. The valuation equilibrium concept we define here reverts to the valuation equilibrium concept devised in Diamantaras and Gilles (1996) when production is conducted through a social production organisation.

Remark 4.3 The first extension of the valuation equilibrium concept seminally considered in Mas-Colell (1980) and Diamantaras and Gilles (1996) to production economies is due to De Simone and Graziano (2004) and Graziano (2007). Our formulation covers these models (Remark 3.5). Furthermore, our notion of valuation equilibrium further generalises the concept used in the cited literature in two main respects. First, in De Simone and Graziano (2004) and Graziano (2007) profit maximization is required under *every* collective good configuration $z \in \mathcal{Z}$, while Definition 4.2 only imposes an optimisation condition at the *realised* collective good configuration z^*, consistent with Diamantaras and Gilles (1996). Second, a collective good configuration $z \in \mathcal{Z}$ directly affects the productive abilities of all agents in the economy. This is not the case for the standard production models considered in the literature.[16] □

4.1 The First Fundamental Theorem of Welfare Economics

Following the literature on extensions of the concept of competitive equilibrium in economies with collective goods—such as Samuelson (1954), Foley (1967, 1970), Kolm (1972), Mas-Colell (1980) and Diamantaras and Gilles (1996)—we pursue the statement of the two welfare theorems for the valuation equilibrium concept for economies with collective goods that are provided through an endogenous social division of labour. The first welfare theorem states that every valuation equilibrium results in a Pareto optimal allocation.

We require an additional property on the indifference sets generated by a preference relation to establish this result. We refer to Hildenbrand (1969) for the seminal discussion of this property.

Assumption 4.4 (Thin indifference sets)
For every agent $a \in A$ we assume that the preference relation $\succsim_a$ has thin indifference sets in the sense that, if $(x, z) \in \mathbb{R}^\ell_+ \times \mathcal{Z}$ with $x \in X_a(z)$ admits a non-empty better set

$$\left\{(x^o, z^o) \in \mathbb{R}^\ell_+ \times \mathcal{Z} \mid x^o \in X_a(z^o) \text{ and } (x^o, z^o) \succ_a (x, z)\right\} \neq \varnothing, \tag{9}$$

then for any $z' \in \mathcal{Z}$ with $(x', z') \succ_a (x, z)$ for some $x' \in X_a(z')$ it holds that

$$\{x^o \in X_a(z') \mid (x^o, z') \succsim_a (x, z)\} \subset \overline{\{x^o \in X_a(z') \mid (x^o, z') \succ_a (x, z)\}} \tag{10}$$

i.e., the weak better set subject to z' is contained in the closure of the better set subject to z'.

The definitions of supporting price-transfer systems and valuation equilibrium allow cases in which valuation equilibria are not supported. Under the assumption on thin indifference sets this is excluded.

Proposition 4.5 *Let $\mathbb{E}$ be an economy such that for almost every agent $a \in A$ her preferences $\succsim_a$ satisfy Assumption 4.4. If (f^*, g^*, c^*, z^*) is a valuation equilibrium in $\mathbb{E}$ for (p, V) such that almost every agent $a \in A$ is non-satiated at $(f^*(a), z^*)$ regarding any collective good configuration $z \in \mathcal{Z}$, then (f^*, g^*, c^*, z^*) is supported by (p, V).*

[16] De Simone and Graziano (2004, Remark 4.7, page 863) remark explicitly that their results extend to the case with z-depending production sets.

For a proof of Proposition 4.5 we refer to Appendix B.

Our statement of the first welfare theorem requires that preferences have thin indifference sets as well as that almost all consumer-producers in the economy are non-satiated at the valuation equilibrium under consideration.

Theorem 4.6 (First Welfare Theorem)
Let $\mathbb{E}$ be an economy such that for almost every agent $a \in A$ her preferences $\succsim_a$ satisfy Assumption 4.4. Let (f^*, g^*, c^*, z^*) be a valuation equilibrium such that almost every $a \in A$ is non-satiated at $(f^*(a), z^*)$ regarding any $z \in \mathcal{Z}$. Then (f^*, g^*, c^*, z^*) is Pareto optimal.

For a proof of this first welfare theorem we refer to Appendix C.

4.2 Supporting Pareto optima by price-valuation systems

We next impose certain assumptions to arrive at a statement of the support of a Pareto optimal configuration by a price system and a valuation system. This is analogous to quasi-equilibrium support results typically established in general equilibrium theory before arriving at the full second welfare theorem. First, we introduce assumptions with regard to the preferences of the consumer-producers and the essentiality of private goods in the economy.

Assumption 4.7 (Essentiality Condition)
Let $\mathbb{E}$ be an economy defined as in Definition 3.4. We assume that for all collective good configurations $z_1, z_2 \in \mathcal{Z}$ with $z_1 \neq z_2$ and every integrable private good allocation $f : A \longrightarrow \mathbb{R}^{\ell}_{+}$ with $f(a) \in X_a(z_1)$ for all $a \in A$, there exists some integrable private good allocation $f' : A \longrightarrow \mathbb{R}^{\ell}_{+}$ such that $f'(a) \in X_a(z_2)$ and $(f'(a), z_2) \succ_a (f(a), z_1)$ for almost all agents $a \in A$.

Our essentiality condition imposes that the preferential losses due to a modification in the collective good configuration can be compensated by the assignment of sufficient levels of private goods. In that regard, the private goods are "essential", and can compensate any change in the collective good configuration in the economy. This condition is weaker than the essentiality condition formulated in Diamantaras and Gilles (1996), since we exclude the case that $z_1 = z_2$ in our formulation.

We remark also that the essentiality condition formulated in Assumption 4.7 implies that all agents are non-satiated in any proposed allocation for any alternative collective good configuration. Additionally, the essentiality condition requires that this non-satiation can be expressed through an integrable private good allocation, which implies that the condition is strictly stronger than non-satiation.

For the statement of the support theorem as well as the second welfare theorem, we have to introduce an assumption on the boundedness of the production correspondence $\mathcal{P}$.

Assumption 4.8 Let $\mathbb{E}$ be an economy defined as in Definition 3.4. We impose on the production correspondence $\mathcal{P}$ in the economy $\mathbb{E}$ the condition that for every collective good configuration $z \in \mathcal{Z}$ there exists an integrable function $\bar{g}_z : A \longrightarrow \mathbb{R}^{\ell}$ such that for every agent $a \in A$: $\mathcal{P}_a(z)$ is bounded from above by $\bar{g}_z(a)$.

Assumption 4.8 implies that the production set has an upper bound and that this upper bound is integrable.

Our first main result asserts that under the formulated assumptions every Pareto optimal allocation can be supported by an appropriately chosen private good price system and valuation system. We refer to this assertion as the Support Theorem in an economy with collective goods provided through an endogenous social division of labour.

Theorem 4.9 (Support Theorem)
Let $\mathbb{E}$ be an economy that satisfies Assumptions 4.4, 4.7 and 4.8. Then every Pareto optimal allocation (f, g, c, z) in $\mathbb{E}$ at which every agent $a \in A$ is non-satiated at $(f(a), z)$ can be supported by a price system $p\colon \mathcal{Z} \to \mathbb{R}^{\ell}_{+} \setminus \{0\}$ and a valuation system $V\colon A \times \mathcal{Z} \to \mathbb{R}$.

For a proof of this theorem we refer to Appendix D of this paper.

The hypotheses under which Pareto optimal allocations can be supported by price-valuation systems are rather weak compared to results stated in the related literature. The main hypothesis on the preferences is that there is non-satiation in the Pareto optimum under consideration. Other hypotheses are mainly technical in nature and do not truly restrict the situations that allow such supporting frameworks to arise. Note that Essentiality assumption 4.7 compares private good allocations for $z_1 \neq z_2$ only, which explicitly necessitates the non-satiation assumption for the case that $z_1 = z_2$.

Next we consider a counterexample to this support theorem with a finite number of agents showing that the non-convexities in the production sets require a continuum of economic agents to resolve.

Example 4.10 A counter example to the support theorem
In finite economies, the result that Pareto optima can be supported through price and valuation systems can no longer be expected to hold due to the non-convexities of the production sets in the economy. In this section we construct a counter-example of a three-agent economy in which there is a Pareto optimal allocation that is not supported by a price vector and a valuation system.
Consider an economy with three agents $A = \{1, 2, 3\}$, a single collective good configuration $\mathcal{Z} = \{z\}$ and two private commodities X and Y. We assume that the collective project is costless, $C(z) = \{0\} = \{(0, 0)\}$.
All agents are characterised by $(X_a, \mathcal{P}_a, \succsim_a)$, where $\succsim_a$ is represented by the utility function $u_a\colon \mathbb{R}^2_+ \to \mathbb{R}$ with $X_a(z) = \mathbb{R}^2_+$, $\mathcal{P}_a(z) = \mathcal{P} = \{(4, 0), (0, 4)\} - \mathbb{R}^2_+$ and $u_a(x, y) = xy$
These utility functions, consumption sets and production sets satisfy Assumptions 4.7 and 4.8. Furthermore, the preferences represented by u_a are strictly convex in the private goods. We show next that, however, there exists a Pareto optimal allocation in this economy that cannot be supported by a price vector and a valuation system.
Consider the allocation $(f, g, 0, z)$ with $f_a = (8/3, 4/3)$ for all a, $g_1 = g_2 = (4, 0) \in \mathcal{P}$ and $g_3 = (0, 4) \in \mathcal{P}$. This allocation is feasible, as can be checked easily.

Claim: $(f, g, 0, z)$ **is Pareto optimal.**
If any agent were to switch production to the other good, there would be no possibility to strictly

increase the utility of at least one agent without reducing the utility of another and maintaining feasibility. If any two agents have unequal consumption bundles, then there would exist a feasible allocation that is a Pareto improvement, found by taking advantage of the differing marginal rates of substitution of the agents with unequal consumption bundles. Hence, the indicated consumption-production plan is indeed Pareto optimal.

Claim: $(f, g, 0, z)$ cannot be supported.
Assume to the contrary that there exists $p(z) = (p, q) \neq (0, 0)$ and $v(z) = V = (V_1, V_2, V_3)$ such that

(i) $V_a \leqslant \max\{4p, 4q\}$ for all $a \in A = \{1, 2, 3\}$;

(ii) $V_1 + V_2 + V_3 = p(z) \cdot c = p(z) \cdot (0, 0) = 0$; and

(iii) For every agent $a \in A$, if $u_a(f', z) \geqslant u_a(f_a, z)$, then $p(z) \cdot f' + V_a \geqslant \max\{4p, 4q\}$.

We first claim that $p = q$.
Indeed, if $p > q$, then $\max\{4p, 4q\} = 4p$ and from (iii) it follows that for every agent $a \in A$: $p \cdot \frac{8}{3} + q \cdot \frac{4}{3} + V_a \geqslant 4p$. Hence,

$$\sum_{a \in A} \left[p \cdot \tfrac{8}{3} + q \cdot \tfrac{4}{3} + V_a \right] = 8p + 4q + \sum_{a \in A} V_a = 8p + 4q \geqslant 12p$$

or $4q \geqslant 4p$, which is a contradiction.
A similar contradiction can be constructed for the case $q > p$. Hence, we conclude that $p = q$.

Furthermore, notice from (ii) that for some agent $b \in A$: $V_b \leqslant 0$. Consider the bundle $f'_b = \left(\frac{32}{15}, \frac{5}{3}\right)$. Then $u_b(f'_b, z) = \frac{32}{15} \cdot \frac{5}{3} = \frac{32}{9} = \frac{8}{3} \cdot \frac{4}{3} = u_b(f_b, z)$.
Hence, from (iii) we conclude now that $p \cdot \frac{32}{15} + q \cdot \frac{5}{3} + V_b \geqslant 4p$. This implies that $p \cdot \left(4 - \frac{32}{15} - \frac{5}{3}\right) = \frac{1}{5}p \leqslant V_b \leqslant 0$. Therefore, $p \leqslant 0$, which is a contradiction. ♦

We point out that if the economy constructed in this example is converted into a continuum economy, the proposed allocation is no longer Pareto optimal. Indeed, exactly half the agents then would produce the first good, exactly half would produce the other, and every agent would consume the bundle $(2, 2)$, which yields higher utility than that obtained by $(8/3, 4/3)$.

4.3 The Second Fundamental Theorem of Welfare Economics

In this section we investigate strengthening the Support Theorem 4.9 to a full statement of the Second Welfare Theorem that Pareto optimal allocations can be supported as valuation equilibria. We show that there are two different statements under slightly different conditions.

We need to strengthen the conditions on the consumptive preferences of the consumer-producers in the economy considered in Assumption 4.4 and Essentiality Condition 4.7. The next assumption brings these conditions together.

Assumption 4.11 Consider a Pareto optimal allocation (f, g, c, z) in an economy $\mathbb{E}$ as defined in Definition 3.4. Then we assume the following properties.

(i) **Upper continuity of preferences at (f, z):**
For every $a \in A$ the preference $\succsim_a$ is *upper continuous* at $(f(a), z)$ in the sense that for every $z' \in \mathcal{Z}$: $\{x' \in X_a(z') \mid (x', z') \succ_a (f(a), z)\}$ is an open set relative to $X_a(z')$.

(ii) **Directional monotonicity at (f, z):**
There exists some $K^* > 0$ and an integrable function $d: A \to \mathbb{R}^\ell_{++}$ such that for almost every $a \in A$: $(f(a) + Kd_a, z) \succ_a (f(a), z)$ for all $0 < K < K^*$.

Assumption 4.11 introduces two new conditions on the preferences and restates two previously discussed properties. Assumption 4.11(i) imposes a standard weak continuity condition on the preferences at a given Pareto optimal allocation.

The directional monotonicity hypothesis 4.11(ii) is a modification of the general differentiability condition introduced in Rubinstein (2012, page 58). The bundle $d_a \gg 0$ represents an improvement bundle, in which direction the preferences are monotonically increasing. Rubinstein shows that, if preferences are *regular*,[17] this directional monotonicity property implies that these preferences can be represented by a differentiable utility function. Clearly, the standard monotonicity conditions discussed in the literature imply this more general directional monotonicity condition.

We note that Assumption 4.11(ii) is weaker than the condition (A.2) of Graziano (2007, page 1043), which imposes that $d_a = d > 0$ for all $a \in A$. Furthermore, Assumption 4.11(ii) is not a consequence of Assumption 4.4 and the non-satiation of the preferences. Indeed, it is not guaranteed that $d_a \gg 0$ by these properties. Finally, Assumption 4.11(ii) imposes certain restrictions on the consumption set correspondence X. In particular, the consumption sets $X_a(z)$ cannot be curved, thin manifolds.

Assumption 4.11 together with the previously stated hypotheses form the requirements for the support of Pareto optimal allocations as a valuation equilibrium. We emphasise that preferences satisfying the properties imposed in Assumption 4.11, even in combination with the assumptions proposed in the previous sections of the paper, can still be incomplete and non-transitive.

Conditions for second welfare theorems. We next investigate the conditions under which the Support Theorem 4.9 can be strengthened to a full version of the Second Welfare Theorem. In the next assumption we introduce conditions imposing that all private commodities are desirable in positive quantities.

Assumption 4.12 Consider a Pareto optimal allocation (f, g, c, z) in an economy $\mathbb{E}$ as defined in Definition 3.4. We assume that for every agent $a \in A$ and every collective good configuration $z' \in \mathcal{Z}$ it holds that $(x', z') \not\succ_a (f(a), z)$ for every $x' \in X_a(z') \cap \partial\mathbb{R}^\ell_+$.

We remark here that if $X_a(z) \subset \mathbb{R}^\ell_{++}$ for all agents $a \in A$ and all collective good configurations $z \in \mathcal{Z}$, Assumption 4.12 is satisfied at any allocation.

We also note that there is no conflict between the hypothesis stated in Assumption 4.12 and the essentiality condition imposed in Assumption 4.7. The condition imposed in Assumption 4.12 requires that any boundary bundle—with at least one commodity not being consumed—cannot be

[17] Regular preferences satisfy the standard neo-classical properties of reflexivity, completeness, transitivity, continuity and (weak) monotonicity.

better than any other available strictly positive consumption bundle. This implies that all goods are desirable for any consumer-producer.

An alternative hypothesis resulting in a statement of the Second Welfare Theorem is formulated using structural concepts from the literature on general equilibrium theory.

Assumption 4.13 Let $\mathbb{E}$ be an economy as defined in Definition 3.4. We impose the following additional conditions on the economy $\mathbb{E}$:

(i) For every collective good configuration $z \in \mathcal{Z}$ there exist an input vector $c \in C(z)$ and some production plan $g: A \to \mathbb{R}^\ell$ such that $g(a) \in \mathcal{P}_a(z)$ for all $a \in A$ and $c \ll \int g \, d\mu$, and;

(ii) **Irreducibility:**
For every Pareto optimal allocation (f, g, c, z) it holds that for every alternative collective good configuration $z' \in \mathcal{Z}$ and for all coalitions $T_1, T_2 \in \Sigma$ with $T_1 \cup T_2 = A$, $T_1 \cap T_2 = \emptyset$, $\mu(T_1) > 0$, and $\mu(T_2) > 0$, there exist an input vector $c' \in C(z')$ and a pair of integrable functions $f': A \to \mathbb{R}^\ell_+$ and $g': A \to \mathbb{R}^\ell$ such that

(a) $g'(a) \in \mathcal{P}_a(z')$ and $f'(a) \in X_a(z')$ for almost every $a \in A$;

(b) $(f'(a), z') \succ_a (f(a), z)$ for every $a \in T_1$, and

(c) $\int_{T_1} f' \, d\mu + c' \leqslant \int_{T_2} g' \, d\mu - \int_{T_2} f' \, d\mu$.

Assumption 4.13(i) requires that the productive capacity in the social division of labour is sufficient to cover the inputs for the creation of an arbitrary collective good configuration as well as a strictly positive allocation of private goods. This assumption is commonly used throughout the literature on general equilibrium in collective good economies, e.g., Diamantaras and Gilles (1996) and Basile, Graziano, and Pesce (2016).

The irreducibility condition in Assumption 4.13(ii) strengthens Essentiality Condition 4.7 by requiring that for every Pareto optimal allocation and every collective good configuration, there exists some allocation of private goods that can improve upon it for any coalition of economic agents, using the productive resources of its complement. This condition was introduced by McKenzie (1959) for exchange economies with a finite number of traders. It was extended to continuum economies by Hildenbrand (1974) and was applied to collective good economies by Graziano (2007).

The next version of the Second Welfare Theorem brings together two different statements under which conditions certain Pareto optimal allocations can be supported as valuation equilibria.

Theorem 4.14 (Second Welfare Theorem)
Let $\mathbb{E}$ be an economy as defined in Definition 3.4 that satisfies Assumptions 4.4, 4.7, 4.8 and 4.11. Then the following statements hold:

(a) *If the economy $\mathbb{E}$ additionally satisfies Assumption 4.12, then every Pareto optimal allocation in $\mathbb{E}$ can be supported as a valuation equilibrium.*

(b) *If the economy $\mathbb{E}$ additionally satisfies Assumption 4.13, then every Pareto optimal allocation in $\mathbb{E}$ can be supported as a valuation equilibrium.*

For a proof of this second welfare theorem we refer to Appendix E of this paper.

5 Increasing returns to specialisation

We investigate in this section the traditional claim that the social division of labour cannot be separated from the idea that human productive ability is subject to learning effects. Gilles (2019b) formalised the hypothesis that human productive abilities are subject to *Increasing Returns to Specialisation* (IRSpec): Specialising in a single productive activity increases productivity in the sense that at least as many units of output can be generated using fewer resources.

Gilles (2019b) shows that the IRSpec property guarantees that there emerges a non-trivial social division of labour in which nearly almost all agents specialise in the production of a single commodity. We investigate a similar claim for an economy with collective goods. We show that most of the insights developed in Gilles (2019b) carry over to the framework that we consider here.

The following definition formalises the notion of Increasing Returns to Specialisation in the context of collective good provision through a social division of labour. As in Gilles (2019b), agents can specialise fully in the production of any single good, which increases the productivity of that agent due to learning effects. The definition of this concept requires two steps. First, the production set of every agent has to be formulated through such full specialisation production plans and, second, these full specialisation production plans are the extreme points in that production set. The following definition formalises the notion of such specialisation and the property that the returns from specialisation are increasing.

Definition 5.1 *Let $\mathbb{E}$ be an economy.*

(i) *For a collective good configuration $z \in Z$, an agent $a \in A$ and a commodity $k \in \{1, \dots, \ell\}$: A production plan $y^k(a,z) \in \mathcal{P}_a(z)$ is a **full specialisation production plan** for commodity k if there exist a strictly positive output level $Q^k(z) > 0$ and an input vector $q^k(z) \in \mathbb{R}^\ell_+$ such that $q^k_k(z) = 0$ and $y^k(a,z) = Q^k(z)\, e_k - q^k(z)$, where $e_k \in \mathbb{R}^\ell_+$ is the k-th unit vector in the commodity space $\mathbb{R}^\ell$.*

(ii) *For an economic agent $a \in A$, the production set correspondence $\mathcal{P}_a$ exhibits **Weakly Increasing Returns to Specialisation** (WIRSpec) if for every collective good configuration $z \in \mathcal{Z}$ the production set $\mathcal{P}_a(z)$ is delimited and for every private good $k \in \{1, \dots, \ell\}$ there exists a full specialisation production plan $y^k(a,z) \in \mathcal{P}_a(z)$ such that*

$$\mathcal{Q}_a(z) \subset \mathcal{P}_a(z) \subseteq \operatorname{Conv} \mathcal{Q}_a(z) - \mathbb{R}^\ell_+ \tag{11}$$

where

$$\mathcal{Q}_a(z) = \left\{ y^k(a,z) \,\middle|\, k = 1, \dots, \ell \right\} \tag{12}$$

is the (finite) set of relevant full specialisation plans and $\operatorname{Conv} \mathcal{Q}_a(z)$ is the convex hull of $\mathcal{Q}_a(z)$.

(iii) *For an economic agent $a \in A$, the production set correspondence $\mathcal{P}_a$ exhibits **Strongly Increasing Returns to Specialisation** (SIRSpec) if $\mathcal{P}_a$ exhibits Weakly Increasing Returns* *to Specialisation (WIRSpec) with respect to* $\mathcal{Q}_a$ *and, additionally, for every collective good configuration* $z \in \mathcal{Z}$: $\mathcal{P}_a(z) \cap \operatorname{Conv} \mathcal{Q}_a(z) = \mathcal{Q}_a(z)$.

(iv) *The economy* $\mathbb{E}$ *satisfies the* ***uniform specialisation property*** *if for every* $a \in A$ *the production correspondence* $\mathcal{P}_a$ *exhibits SIRSpec for* $\mathcal{Q}_a = \mathcal{Q}$*, where for every collective good configuration* $z \in \mathcal{Z}$ *and every commodity* $k \in \{1, \dots, \ell\}$*, there exists a unique full specialisation production plan* $\hat{y}^k(z) = Q^k(z)e_k - q^k(z) \in \mathbb{R}^\ell$ *such that* $\mathcal{Q}(z) = \left\{ \hat{y}^k(z) \,\middle|\, k = 1, \dots, \ell \right\}$.

The property of *Increasing Returns to Specialisation* (IRSpec) represents that specialising in a single output leads to learning effects and increased productivity. Technically, this is represented by two related mathematical properties. The weak version of IRSpec imposes that there exist ℓ full specialisation production plans—one for each of the ℓ commodities—that form the outermost corner points of the production set.[18]

The second property introduces a strong version of IRSpec, which additionally imposes that the outermost plans in the production set are exactly the ℓ constructed full specialisation production plans. The SIRSpec property was also introduced by Gilles (2019b) and has been shown to form the foundation of the emergence of a proper social division of labour in the prevailing equilibrium. Below we extend this insight to our framework through the valuation equilibrium concept.

The uniform specialisation property strengthens Increasing Returns to Specialisation in that the SIRSpec property assumed to be applicable to all agents in the economy for a given common set of full specialisation production plans. The collectively available full specialisation production plans are such that all agents have access to Increasing Returns to Specialisation, based on specialised abilities that can be acquired through the collective education system. Therefore, this property refers to an economy in which there is some institutional framework of knowledge sharing in which all agents have access to the same production technologies and are able to achieve the same productivity level if fully specialised in the production of a single commodity.

It is clear that, to achieve uniform specialisation, any collective education and training system to enact such knowledge sharing can be considered as part of the collective good configuration $z \in \mathcal{Z}$. Various investment levels in such a collective education system can be represented by the levels of private good investments $c \in C(z)$ that are required. Different configurations of the education system can result in different trained productive abilities represented by the full specialisation production plans $\hat{y}^k(z)$ for $k \in \{1, \dots, \ell\}$.

The main consequence of the uniform specialisation property is that all equilibrium prices in a valuation equilibrium are completely determined by the full specialisation production plans that are the corner points of every agent's production set for any collective good configuration.

Theorem 5.2 *Let* $\mathbb{E}$ *be an economy that satisfies Assumption 4.4 and let* (f^*, g^*, c^*, z^*) *be a valuation equilibrium with* (p, V)*. Furthermore, assume that every* $a \in A$ *is non-satiated at every* $(f^*(a), z^*)$.

(a) *Suppose that for every* $a \in A$ *the production correspondence* $\mathcal{P}_a$ *exhibits WIRSpec for* $\mathcal{Q}_a$*. Then for every* $a \in A$: $p(z^*) \cdot g^*(a) = \max\, p(z^*) \cdot \mathcal{Q}_a(z^*)$.

[18] The property as formulated in Definition 5.1(ii) above is similar to what is referred to as the property of Weak Increasing Returns to Specialisation (WIRSpec) in Gilles (2019b).

(b) *Suppose that for every* $a \in A$ *the production correspondence* $\mathcal{P}_a$ *exhibits SIRSpec for* Q_a*. Then, if* $p(z^*) \gg 0$*, it holds that for every* $a \in A$: $g^*(a) \in Q_a(z^*)$*, inducing a non-trivial social division of labour given by* $\{A_1(g^*), \dots, A_\ell(g^*)\}$ *where*

$$A_k(g^*) = \{a \in A \mid g^*_k(a) > 0\} \in \Sigma \qquad \text{for } k \in \{1, \dots, \ell\}.$$

The collection $\{A_1(g^*), \dots, A_\ell(g^*)\}$ *forms a partitioning of* A.

(c) *Suppose that the economy* $\mathbb{E}$ *satisfies the uniform specialisation property for* $Q(z)$, $z \in \mathcal{Z}$*. If* $p(z^*) \gg 0$ *and* $\int g^* \, d\mu \gg 0$*, then* $p(z^*)$ *is characterised by the equation system*

$$p(z^*) \cdot \hat{y}^k(z^*) = p(z^*) \cdot \hat{y}^m(z^*) \quad \text{for all } k, m \in \{1, \dots, \ell\}. \tag{13}$$

For a proof of Theorem 5.2 we refer to Appendix F.

Theorem 5.2 investigates the properties of the equilibrium for the three different properties that introduce Increasing Returns to Specialisation into the setting of an economy with collective goods.

Under the basic hypothesis of WIRSpec, we establish that the incomes generated in equilibrium are the same as achieved by those full specialisation production plans that attain maximal incomes. Hence, equilibrium production generates income levels that are achieved under full specialisation.

Strengthening the relevant property to SIRSpec essentially establishes that in every valuation equilibrium there emerges a true and non-trivial social division of labour.

If the idea of Increasing Returns to Specialisation is extended to mean that there is some set of ℓ common "professions"—each represented by a fixed full specialisation production plan—then there is complete equality in any valuation equilibrium in which all private goods are produced in non-negligible quantities. In particular, all objectively given professions generate exactly the same income level in that equilibrium. Such income equalisation implies that the equilibrium prices are exactly equal to the corresponding Leontief prices (Leontief, 1936).

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Appendix: Proofs

A Proofs of Section 2

A.1 Proof of Proposition 2.1

Consider the allocation (f^*, g^*, z^*) given by

$$f^*(a) = \left(\tfrac{1}{2}, 0\right),$$
$$g^*(a) = \begin{cases} (1,0) & \text{if } a \leqslant \tfrac{1}{2}, \\ (0,1) & \text{if } a > \tfrac{1}{2}, \end{cases}$$
$$z^* = \tfrac{1}{2}.$$

We claim that (f^*, g^*, z^*) is Pareto optimal.
Indeed, first note that (f^*, g^*, z^*) is feasible, since $\int f^* d\mu + c(z^*) = \left(\frac{1}{2}, 0\right) + \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right) = \int g^* \, d\mu$. Second, assume that there exists some feasible allocation (f, g, c, z) in this economy such that for almost all economic agents $a \in A$:

$$u_a\left(f(a), z\right) = f_x(a) \cdot z \geqslant u_a\left(f^*(a), z^*\right) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4},$$

and for some coalition $S \in \Sigma$ with $\mu(S) > 0$ it holds that for all $a \in S$:

$$u_a\left(f(a), z\right) = f_x(a) \cdot z > u_a\left(f^*(a), z^*\right) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4},$$

where we use the notation $f(a) = \big(f_x(a), f_y(a)\big)$.
We remark that this implies that $z > 0$ as well as that

$$\int f_x \, d\mu > \frac{1}{4z}. \tag{14}$$

We now introduce the coalitions given by

$$\begin{aligned} T &= \{a \in A \mid g(a) \leqslant (0,0)\}, \\ T_x &= \{a \in A \mid g(a) \leqslant (1,0) \text{ and } g_x(a) > 0\}, \\ T_y &= \{a \in A \mid g(a) \leqslant (0,1) \text{ and } g_y(a) > 0\}. \end{aligned}$$

Note that $\mu(T_x) + \mu(T_y) \leqslant 1$, since $A = T \cup T_x \cup T_y$. Furthermore, from feasibility,

$$\int f \, d\mu + (0,z) = \int g \, d\mu \leqslant \big(\mu(T_x), \mu(T_y)\big) .$$

Hence,

$$\int f_x \, d\mu + \int f_y \, d\mu + z \leqslant \mu(T_x) + \mu(T_y) \leqslant 1. \tag{15}$$

Now, from (14) and (15) we conclude that[19]

$$1 > \tfrac{1}{4z} + \int f_y \, d\mu + z \geqslant \tfrac{1}{4z} + z \geqslant 2\sqrt{\tfrac{1}{4z} z} = 1$$

which constitutes a contradiction. Thus, we have shown that (f^*, g^*, z^*) is indeed Pareto optimal.

Next, we claim that (f^*, g^*, z^*) *can be supported by the price system given by* $p(z) = (1,1)$ *for all* $z \in \mathcal{Z}$ *and a tax system given by* $V(a,z) = z$ *for all* $a \in A$ *and* $z \in \mathcal{Z} = [0,1]$.
First, note that for all policing levels $z \in \mathcal{Z}$: $p(z) \cdot (1,0) = p(z) \cdot (0,1) = 1$, so the generated income from production $\sup p(z) \cdot \mathcal{P}_a(z) = \max p(z) \cdot \mathcal{P}_a(z) = 1$ for all $a \in A$. Therefore, $V(a,z) = z \leqslant 1$. Furthermore, for any $z \in \mathcal{Z}$ we have budget balance with

$$\int V(\cdot, z) \, d\mu = z = (1,1) \cdot (0,z) = p(z) \cdot c(z)$$

Next, we note that $(f^*(a), g^*(a), z^*)$ is indeed an element of a's budget set, since

$$p(z^*) \cdot f^*(a) + V(a, z^*) = (1,1) \cdot \left(\tfrac{1}{2}, 0\right) + \tfrac{1}{2} = 1 = p(z^*) \cdot g^*(a) = \sup p(z^*) \cdot \mathcal{P}_a(z^*).$$

To show that $(f^*(a), g^*(a), z^*)$ is a u_a-maximiser in a's budget set, assume to the contrary that there exists some (f'', g'', z'') such that (A) $f''_x z'' > \frac{1}{4}$ and (B) $p(z'') \cdot f'' + V(a, z'') \leqslant p(z'') \cdot g''$. That is, from (B), $f''_x + f''_y + z'' \leqslant 1$. Since $f''_y \geqslant 0$ it holds that $f''_x + z'' \leqslant 1$. Now combining with (A), it follows that $(1 - z'')z'' > \frac{1}{4}$, which is equivalent to $(2z'' - 1)^2 < 0$. This is impossible, implying individual optimality.

[19] The argument is based on the fact that for $a, b \geqslant 0$: $a + b \geqslant 2\sqrt{ab}$, derived from $(\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab} \geqslant 0$.

A.2 Proof of Proposition 2.2

Consider the allocation (f^*, g^*, γ^*) defined as follows

$$\begin{cases} f^*(a) = \left(\frac{3}{4}, \frac{3\Gamma}{2}\right) & g^*(a) = (1,0) \qquad \gamma^*(a) = 0 \qquad \text{for } a \leqslant \frac{1}{2} \\ f^*(a) = \left(\frac{1}{4}, \frac{\Gamma}{2}\right) & g^*(a) = (0, \gamma^*(a)) \qquad \gamma^*(a) = 2\Gamma \qquad \text{for } a > \frac{1}{2} \end{cases}$$

Note first that (f^*, g^*, γ^*) is balanced: $\int f^*(a)\, da + c(\gamma^*) = \int g^*(a)\, da = \left(\frac{1}{2}, \Gamma\right)$.
Furthermore, for any $a \in [0,1]$: $\max p(\gamma) \cdot \mathcal{P}_a(\gamma) = \max\{2\Gamma, \gamma(a)\} = 2\Gamma$.
Therefore, for all $a \leqslant \frac{1}{2}$ and $\gamma \in \mathcal{Z}$ we have that $\max p(\gamma) \cdot \mathcal{P}_a(\gamma) = 2\Gamma > V(a,\gamma) = -\Gamma$ as well as for $a > \frac{1}{2}$ it holds that $\max p(\gamma) \cdot \mathcal{P}_a(\gamma) = 2\Gamma > V(a,\gamma) = \Gamma$.
We further remark that in equilibrium there is indeed budget neutrality, since

$$\int V(a, \gamma^*)\, da = -\tfrac{1}{2}\,\Gamma + \tfrac{1}{2}\,\Gamma = 0 = p(\gamma^*) \cdot c(\gamma^*).$$

Also, for $\gamma \neq \gamma^*$: $\int V(a,\gamma)\, da = 0 = p(\gamma) \cdot c(\gamma)$.

To complete the proof, assume that for some $a \in A$ there exists (f, g, γ) with $f = (x, y)$ and $g \in \{(1,0), (0, \gamma_a)\}$ such that $u(x,y) = xy > u(f^*(a))$ and $p(\gamma) \cdot (x,y) + V(a,\gamma) \leqslant p(\gamma) \cdot g \leqslant \max p(\gamma) \cdot \mathcal{P}_a(\gamma) = 2\Gamma$.
For every $\gamma \in [0, 2\Gamma\,]$ it now follows that:

- for $a \leqslant \frac{1}{2}$: $xy > \frac{9\Gamma}{8}$ and $2\Gamma x + y - \Gamma \leqslant 2\Gamma$. Hence, $x > \frac{9\Gamma}{8y}$ as well as $\frac{9\Gamma^2}{4y} + y < 3\Gamma$. Hence, $4y^2 - 12\Gamma y + 9\Gamma^2 = (2y - 3\Gamma)^2 < 0$, which is an impossibility.
- for $a > \frac{1}{2}$: $xy > \frac{\Gamma}{8}$ and $2\Gamma x + y + \Gamma \leqslant 2\Gamma$. Hence, $0 \leqslant (\sqrt{2\Gamma x} - \sqrt{y})^2 = 2\Gamma x + y - 2\sqrt{2\Gamma xy} < \Gamma - 2\sqrt{2\Gamma \cdot \frac{\Gamma}{8}} = 0$, which is an impossibility as well.

This completes the proof of the assertion stated in the proposition.

A.3 Proof of Proposition 2.3

Consider $0 \leqslant \lambda \leqslant 1$ and define the allocation $A_\lambda = (f^*, g^*, \lambda, z^*)$ as stated in Proposition 2.3. Note that allocation A_λ is generated through $\mu_x = \frac{5}{12} + \frac{1}{4}\lambda$ and $\mu_y = \frac{7}{12} - \frac{1}{4}\lambda$.
Also, for all $a \in A$: $I(a, z^*) = \max p(z^*) \cdot \mathcal{P}_a(z^*) = \frac{1}{3} = z^*$ and $f^*(a) = \left(\frac{1}{3}, \frac{1}{9}\right) = \left(\frac{z^* - V(a,z^*)}{2z^*}, \frac{z^* - V(a,z^*)}{2}\right)$.
As a consequence, A_λ is a feasible allocation.

We now investigate that (p, V) indeed supports A_λ:

(i) For every $a \in A$ and $z \in [0,1]$: $V(a,z) = z^2 \leqslant z = \max p(z) \cdot \mathcal{P}_a(z)$ and $\int V(a,z)\, da = z^2 = p(z) \cdot c$ for any $c \in C(z)$.

(ii) Finally, assume that there exists some $a \in A$ and $f = (x,y)$ with $u_a(x,y) > u_a(f^*(a)) = \frac{1}{27}$ and $p(z) \cdot f + V(a,z) \leqslant \max p(z) \cdot \mathcal{P}_a(z) = z$. That is, $xy > \frac{1}{27}$ as well as $zx + y + z^2 \leqslant z$. Hence, we arrive at

$$\frac{z}{27y} + y + z^2 < z \quad \text{implying} \quad 27y^2 + z(z-1)27y + z < 0.$$

This quadratic equation in y has no solution because $z \in [0, 1]$. This is a contradiction.[20]

This shows that, indeed, the derived configuration is an equilibrium.

B Proof of Proposition 4.5

Let (f^*, g^*, c^*, z^*) be a valuation equilibrium for the price system $p\colon \mathcal{Z} \to \mathbb{R}^\ell_+ \setminus \{0\}$ and valuation system $V\colon A \times \mathcal{Z} \to \mathbb{R}$.
Take any agent $a \in A$ such that there exist $z \in \mathcal{Z}$ and $f \in X_a(z)$ with $(f, z) \succsim_a (f^*(a), z^*)$. We need to show that

$$p(z) \cdot f + V(a, z) \geqslant \sup p(z) \cdot \mathcal{P}_a(z).$$

Since $a \in A$ is non-satiated at $(f^*(a), z^*)$ regarding any z, there exists some $x \in X_a(z)$ such that $(x, z) \succ_a (f^*(a), z^*)$. That is,

$$\left\{(x, z) \in \mathbb{R}^\ell_+ \times \mathcal{Z} \,\middle|\, x \in X_a(z) \text{ and } (x, z) \succ_a (f^*(a), z^*)\right\} \neq \varnothing.$$

Hence, by Assumption 4.4,

$$f \in \{x \in X_a(z) \mid (x, z) \succsim_a (f^*(a), z^*)\} \subset \overline{\{x \in X_a(z) \mid (x, z) \succ_a (f^*(a), z^*)\}}.$$

Thus, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(x_n, z) \succ_a (f^*(a), z^*)$ with $x_n \to f$. Since (f^*, g^*, c^*, z^*) is a valuation equilibrium, $(x_n, g, z) \notin B(a, p, V)$ for any production plan $g \in \mathcal{P}_a(z)$. Therefore, for every $g \in \mathcal{P}_a(z)$: $p(z) \cdot x_n + V(a, z) > p(z) \cdot g$ implying that $p(z) \cdot x_n + V(a, z) \geqslant \sup p(z) \cdot \mathcal{P}_a(z)$. Hence, taking $n \to \infty$ implies now that $p(z) \cdot f + V(a, z) \geqslant \sup p(z) \cdot \mathcal{P}_a(z)$, which proves the assertion.

C Proof of Theorem 4.6

Let (f^*, g^*, c^*, z^*) be a valuation equilibrium with equilibrium price system $p\colon \mathcal{Z} \to \mathbb{R}^\ell_+ \setminus \{0\}$ and valuation system $V : A \times \mathcal{Z} \to \mathbb{R}$.
Assume that (f^*, g^*, c^*, z^*) is *not* Pareto optimal. Then there exists an alternative feasible allocation (f, g, c, z) such that $f(a) \in X_a(z)$, $g(a) \in \mathcal{P}_a(z)$, $c \in C(z)$ and $(f(a), z) \succsim_a (f^*(a), z^*)$ for almost all $a \in A$ and $(f(b), z) \succ_b (f^*(b), z^*)$ for all $b \in S \subseteq A$, with $\mu(S) > 0$. Then, by definition of valuation equilibrium , it holds that

$$p(z) \cdot f(b) + V(b, z) > p(z) \cdot g(b) \qquad \text{for all } b \in S.$$

Since, by assumption, all $a \in A$ are non-satiated at $(f^*(a), z^*)$ regarding z in the sense that there exists some $x \in X_a(z)$ with $(x, z) \succ_a (f^*(a), z^*)$, it follows by Assumption 4.4 and a similar argument as used in the proof of Proposition 4.5, it follows that $p(z) \cdot f(a) + V(a, z) \geqslant p(z) \cdot g(a)$ for almost every $a \in A \setminus S$.
This, in turn, implies that $p(z) \cdot \int f \, d\mu + \int V(\cdot, z) \, d\mu > p(z) \cdot \int g \, d\mu$, which, by condition (iii) of Definition 4.1, yields in particular that $p(z) \cdot \int f \, d\mu + p(z) \cdot c > p(z) \cdot \int g \, d\mu$, contradicting the feasibility of (f, g, c, z).

[20] Indeed, the equation's determinant is non-positive: $\Delta = 729z(z - \frac{1}{3})^2(z - \frac{4}{3}) \leqslant 0$. We remark here that, in particular, $\Delta = 0$ if and only if $z = 0$ or $z = z^* = \frac{1}{3}$, confirming the feasibility of the equilibrium configuration.

D Proof of Theorem 4.9

Let (f, g, c, z) be a Pareto optimal allocation at which all agents are non-satiated. Define for every agent $a \in A$ and some arbitrary collective good configuration $z' \in \mathcal{Z}$ the following sets

$$F(a, z') = \{x \in X_a(z') \mid (x, z') \succ_a (f(a), z)\} \tag{16}$$

$$F(z') = \int F(\cdot, z')\, d\mu + C(z') - \int \mathcal{P}(\cdot, z')\, d\mu \tag{17}$$

Since every $a \in A$ is non-satiated at $(f(a), z)$, it follows that $F(a, z) \neq \emptyset$ and, consequently, from Assumption 4.4 it follows that $f(a) \in \{x \in X_a(z) \mid (x, z) \succsim_a (f(a), z)\} \subset \overline{F(a, z)}$. This implies that there is some $x \in F(a, z)$ with $x \leqslant f(a) + e$, where $e = (1, 1, \dots, 1) \gg 0$. Hence,

$$G(a, z) = F(a, z) \cap \{x \in \mathbb{R}^\ell_+ \mid 0 \leqslant x \leqslant f(a) + e\} \neq \emptyset.$$

Clearly, $G(\cdot, z)$ has a measurable graph due to the assumed measurability of $\succ_a$ in the definition of $\mathbb{E}$ and is integrably bounded by $f + e$. Thus, by Aumann's measurable selection theorem and integrably boundedness of that selection, $G(\cdot, z)$ has an integrable selection, implying in turn that $F(\cdot, z)$ has an integrable selection. Therefore, $\int F(\cdot, z)\, d\mu \neq \emptyset$.
For $z' \neq z$, Essentiality Condition 4.7 guarantees that $F(a, z') \neq \emptyset$ and has an integrable selection. This implies that $\int F(\cdot, z')\, d\mu \neq \emptyset$.

From the definition of $\mathbb{E}$, it is imposed that $C(z') \neq \emptyset$ and the correspondence $\mathcal{P}(\cdot, z')$ has a measurable graph. Moreover, from Assumption 4.8 it follows that $\mathcal{P}(\cdot, z')$ is integrably bounded from above, implying that $\mathcal{P}(\cdot, z')$ actually has an integrable selection. Hence, $\int \mathcal{P}(\cdot, z')\, d\mu \neq \emptyset$.

Therefore, combined with the above, we conclude that $F(z') \neq \emptyset$ for every $z' \in \mathcal{Z}$.

Lemma D.1 *For every $z' \in \mathcal{Z}$: $F(z') \cap \mathbb{R}^\ell_- = \emptyset$.*

Proof. Assume by way of contradiction the existence of $x \in F(z') \cap \mathbb{R}^\ell_-$ for some $z' \in \mathcal{Z}$. This means that x can be rewritten as

$$x = \int f'\, d\mu + c' - \int g'\, d\mu \leqslant 0,$$

with $c' \in C(z')$, $f'(a) \in F(a, z')$ and $g'(a) \in \mathcal{P}_a(z')$ for almost all $a \in A$. Free disposal in production allows us to select g' such that $x = 0$. Thus, (f', g', c', z') is a feasible allocation that forms a Pareto improvement of (f, g, c, z). This is a contradiction. ∎

Lemma D.2 *For every $z' \in \mathcal{Z}$ there exists some $p(z') > 0$ such that $p(z') \cdot x \geqslant 0$ for all $x \in F(z')$.*

Proof. Let $z' \in \mathcal{Z}$ be some collective good configuration. By hypothesis, $C(z')$ is convex and, thus by Lyapunov's Theorem, $F(z')$ is convex (Hildenbrand, 1974, Theorem 3, page 62). Therefore, from Lemma D.1, Minkowski's separation theorem (Hildenbrand, 1974, (11), page 38) applies and there exists some $p(z') \neq 0$ such that

$$\sup_{y \in \mathbb{R}^\ell_-} p(z') \cdot y \leqslant \inf_{x \in F(z')} p(z') \cdot x. \tag{18}$$

We now show that $p(z') > 0$. Suppose there is some $h \in \{1, \dots, \ell\}$ with $p_h(z') < 0$. Then for any $Q > 0$ select $y(Q) \in \mathbb{R}^\ell_-$ by $y_k(Q) = 0$ if $k \neq h$ and $y_h(Q) = -Q$. Then $p(z') \cdot y(Q) = -p_h(z')\, Q > 0$

and can be made arbitrarily large by selecting large enough $Q > 0$. Hence, $\sup_{y\in\mathbb{R}^{\ell}_{-}} p(z')\cdot y = \infty$. On the other hand, $F(z') \neq \emptyset$ implies that $\inf p(z')\cdot F(z') < \infty$ is finite. These two conclusions contradict the Minkowski inequality (18).
Thus, we conclude that $p(z') > 0$.
Furthermore, $p(z') > 0$ implies that $\sup_{y\in\mathbb{R}^{\ell}_{-}} p(z')\cdot y = 0$, leading to the conclusion that $p(z')\cdot x \geqslant 0$ for all $x \in F(z')$. ■

Define for all $a \in A$ and every $z' \in \mathcal{Z}$,

$$t(a,z') := \inf\,\{p(z')\cdot x \mid x \in F(a,z')\}.$$

Since $F(a,z') \neq \emptyset$ is bounded from below by zero, $t(a,z') \geqslant 0$ for almost all $a \in A$.
The function $t(\cdot,z')\colon A \to \mathbb{R}_+$ is measurable, since by the definition of $\mathbb{E}$, $F(\cdot,z')$ has a measurable graph and Hildenbrand (1974, Proposition 3, page 60) applies.
Furthermore, $\int F(\cdot,z')\,d\mu \neq \emptyset$ as shown above, implies that there is some integrable selection f' of $F(\cdot,z')$. By definition, $0 \leqslant t(\cdot,z') \leqslant p(z')\cdot f'(a)$. Hence, $t(\cdot,z')$ is integrably bounded, and therefore integrable.
Finally, note that since every $a \in A$ is non-satiated at $(f(a),z)$, from Assumption 4.4 and a similar argument as used in the proof of Proposition 4.5, it follows that

$$p(z)\cdot f(a) \geqslant t(a,z). \tag{19}$$

Define for every $a \in A$ and every $z' \in \mathcal{Z}$,

$$I(a,z') := \sup\, p(z')\cdot \mathcal{P}_a(z') \geqslant 0.$$

Now, the function $I(\cdot,z')\colon A \to \mathbb{R}_+$ is measurable, since by the definition of $\mathbb{E}$, $\mathcal{P}(\cdot,z')$ has a measurable graph and Hildenbrand (1974, Proposition 3, page 60) applies. Moreover, $I(\cdot,z')$ is integrable, since from Assumption 4.8, $\mathcal{P}(\cdot,z')$ is integrably bounded from above.
Also, Proposition 6 in Hildenbrand (1974, page 63) guarantees that

$$\inf p(z')\cdot F(z') = \int \inf\,\{p(z')\cdot x \mid x \in [F(a,z') + C(z') - \mathcal{P}_a(z')\,]\}\;d\mu$$

Therefore, since $p(z')\cdot x \geqslant 0$ for every $x \in F(z')$, it follows that $\inf p(z')\cdot F(z') \geqslant 0$, implying that

$$\inf\, p(z')\cdot F(z') = \int t(\cdot,z')\,d\mu + \inf p(z')\cdot C(z') - \int I(\cdot,z')\,d\mu \geqslant 0. \tag{20}$$

We can now define a valuation system $V\colon A\times\mathcal{Z} \to \mathbb{R}$ by $V(a,z') := I(a,z') - t(a,z')$ for every agent $a \in A$ and every collective good configuration $z' \in \mathcal{Z}$. Evidently, the valuation system $V(\cdot,z')$ is integrable on (A,Σ,μ) for every $z' \in \mathcal{Z}$.

Next we prove that (f,g,c,z) is a supported through the valuation system V and the price system p constructed above.

(i) From the above, for every agent $a \in A$ and every collective good configuration $z' \in \mathcal{Z}$, $t(a,z') = \inf\, p(z')\cdot F(a,z') \geqslant 0$. Therefore, we may conclude that $V(a,z') = I(a,z') - t(a,z') \leqslant I(a,z') = \sup p(z')\cdot \mathcal{P}_a(z')$.

(ii) We now prove the budget neutrality for the constructed price-valuation system. From feasibility of (f, g, c, z) and (19) it follows that

$$\int V(\cdot, z)\, d\mu = \int [\, I(\cdot, z) - t(\cdot, z)\,]\; d\mu \geqslant p(z) \cdot \int (g - f)\, d\mu = p(z) \cdot c.$$

On the other hand, from (20) it follows that

$$\int V(\cdot, z)\, d\mu = \int I(\cdot, z)\, d\mu - \int t(\cdot, z)\, d\mu \leqslant \inf p(z) \cdot C(z) \leqslant p(z) \cdot c.$$

Hence the assertion is shown.

(iii) The collective good configuration z maximises the surplus. Indeed, from (20) it follows that for every $z' \neq z$: $\int V(\cdot, z')\, d\mu = \int I(\cdot, z')\, d\mu - \int t(\cdot, z')\, d\mu \leqslant \inf p(z') \cdot C(z') \leqslant p(z') \cdot c'$ for any $c' \in C(z')$, confirming the assertion.

(iv) Finally, we show that for every agent $a \in A$ and (f', z') with $f' \in X_a(z')$, $(f', z') \gtrsim_a (f(a), z)$ implies that $p(z') \cdot f' + V(a, z') \geqslant \sup p(z') \cdot \mathcal{P}_a(z')$.
In particular, $F(a, z') \neq \varnothing$ and, therefore, by Assumption 4.4 and a similar argument as used in the proof of Proposition 4.5, it follows that $p(z') \cdot f' + V(a, z') \geqslant \sup p(z') \cdot \mathcal{P}_a(z')$.
This shows the desired assertion.

From (i)–(iv) we conclude now that (f, g, c, z) is indeed supported by the price system p and the valuation system V, showing Theorem 4.9.

E Proof of Theorem 4.14

Suppose that $\mathbb{E}$ is an economy as defined in Definition 3.4 that satisfies Assumptions 4.4, 4.7, 4.8 and 4.11. Now, let (f, g, c, z) be a Pareto optimal allocation in $\mathbb{E}$.
By these hypotheses and the fact that the directional monotonicity Assumption 4.11(ii) implies non-satiation, the assertion of Theorem 4.9 holds and guarantees that (f, g, c, z) can be supported by a positive price system p and a valuation system V. We recall from the proof of Theorem 4.9 that for every $a \in A$ and $z' \in \mathcal{Z}$,

$$\begin{aligned} I(a, z') &= \sup p(z') \cdot \mathcal{P}_a(z') \geqslant 0 \\ t(a, z') &= \inf p(z') \cdot F(a, z') \geqslant 0 \\ V(a, z') &= I(a, z') - t(a, z'), \end{aligned}$$

where $F(a, z')$ is as defined in (16).
The only property remaining to show is that for every $a \in A$, the triple $(f(a), g(a), z)$ optimises a's consumptive preferences $\gtrsim_a$ on the budget set $B(a, p, V)$.

First we notice that for every agent $a \in A$, the triple $(f(a), g(a), z) \in B(a, p, V)$. To this end we need the following Lemma.

Lemma E.1 *Under the assumptions stated, for almost every $a \in A$ it holds that $I(a, z) = p(z) \cdot g(a) = \sup\, p(z) \cdot \mathcal{P}_a(z)$ and $t(a, z) = p(z) \cdot f(a)$.*

Proof. Assume to the contrary that there exists some coalition S with $\mu(S) > 0$ such that $I(a, z) > p(z) \cdot g(a)$ for all $a \in S$. Then for all $a \in S$ there exists $h(a) \in \mathcal{P}_a(z)$ such that $p(z) \cdot h(a) > p(z) \cdot g(a)$. Hence, we may define a correspondence $\Phi \colon S \to 2^{\mathbb{R}^\ell}$ given by

$$\Phi(a) = \{g \in \mathcal{P}_a(z) \mid p(z) \cdot g(a) < p(z) \cdot g \leqslant \sup p(z) \cdot \mathcal{P}_a(z)\} \neq \varnothing.$$

The correspondence Φ has a measurable graph and is integrably bounded by Assumption 4.8. Therefore, there exists some integrable selection $\bar{g}\colon S \to \mathbb{R}^\ell$ in Φ.
Now, for every $a \in S$ it holds that

$$K = \min\left\{\frac{1}{p(z)\cdot\int d\,d\mu}\int_S [p(z)\cdot\bar{g}(a) - p(z)\cdot g(a)]\,d\mu\,,\ K^*\right\} > 0,$$

where $d\colon A \to \mathbb{R}^\ell_{++}$ assigns to every $a \in A$ the direction $d_a \gg 0$ in which $\succsim_a$ is increasing (Assumption 4.11(ii)) and $K^* > 0$ is the uniform bound on this increase.
Now, define the private goods allocation f' by $f'(a) = f(a) + \frac{1}{2}K\,d_a$ for almost all $a \in A$, and the production plan $g'(a) = g(a)$ for $a \in A \setminus S$ and $g'(a) = \bar{g}(a)$ for $a \in S$.
From the directional monotonicity of $\succsim_a$ in $(f(a), z)$ and the fact that $\frac{1}{2}K < K^*$ we note that $f'(a) \in F(a,z)$ and, therefore, $x' = \int f'\,d\mu + c - \int g'\,d\mu \in F(z)$, implying $p(z)\cdot x' \geqslant 0$.
On the other hand, by feasibility of (f, g, c, z),

$$\begin{aligned}
p(z)\cdot x' &= p(z)\cdot\int f'\,d\mu + p(z)\cdot c - p(z)\cdot\int g'\,d\mu = \\
&= p(z)\cdot\int f\,d\mu + \tfrac{1}{2}K\,p(z)\cdot\int d\,d\mu + p(z)\cdot c - p(z)\cdot\int g'\,d\mu < \\
&< p(z)\cdot\int f\,d\mu + K\,p(z)\cdot\int d\,d\mu + p(z)\cdot c - p(z)\cdot\int g'\,d\mu \leqslant \\
&\leqslant p(z)\cdot\int f\,d\mu + p(z)\cdot\int_S \bar{g}\,d\mu - p(z)\cdot\int_S g\,d\mu \\
&\qquad\qquad + p(z)\cdot c - p(z)\cdot\int_S \bar{g}\,d\mu - p(z)\cdot\int_{A\setminus S} g\,d\mu = \\
&= p(z)\cdot\int f\,d\mu + p(z)\cdot c - p(z)\cdot\int g\,d\mu = 0.
\end{aligned}$$

This is a contradiction showing the first assertion of the lemma.

To show the second assertion, we note first that for all $a \in A$: $t(a,z) \leqslant p(z)\cdot f(a)$ by (19).
Next, assume to the contrary that there exists some coalition $S \in \Sigma$ with $\mu(S) > 0$ and $t(a,z) < p(z)\cdot f(a)$ for all $a \in S$. From the feasibility of (f, g, c, z) it then follows that

$$\int t(\cdot,z)\,d\mu + p(z)\cdot c < \int p(z)\cdot f\,d\mu + p(z)\cdot c = \int p(z)\cdot g\,d\mu \leqslant \int I(\cdot,z)\,d\mu.$$

Hence, $\int V(\cdot,z)\,d\mu = \int I(\cdot,z)\,d\mu - \int t(\cdot,z)\,d\mu > p(z)\cdot c$.
This contradicts the budget neutrality condition stated as Definition 4.1(ii) shown in the proof of Theorem 4.9. ■

Finally, since by Lemma E.1 it holds that $I(a,z) = p(z)\cdot g(a)$ and $t(a,z) = p(z)\cdot f(a)$, it follows that $p(z)\cdot f(a) + V(a,z) = \sup\, p(z)\cdot\mathcal{P}_a(z) = p(z)\cdot g(a)$, indeed showing that $(f(a), g(a), z) \in B(a,p,V)$.

Furthermore, we show that the main assertion holds for bundles that have positive value under the prevailing prices:

Lemma E.2 *Suppose that for agent $a \in A$ there is some (f', z') such that $f' \in X_a(z')$, $(f', z') \succ_a (f(a), z)$ and $p(z')\cdot f' > 0$. Then $p(z')\cdot f' + V(a,z') > I(a,z') \geqslant p(z')\cdot g'$ for every $g' \in \mathcal{P}_a(z')$.*

Proof. Note that, since (f, g, c, z) is supported by (p, V), from the property that $I(a,z') = \sup\, p(z')\cdot\mathcal{P}_a(z')$ it follows that $p(z')\cdot f' + V(a,z') \geqslant I(a,z') \geqslant p(z')\cdot g'$ for all $g' \in \mathcal{P}_a(z')$.

Now suppose to the contrary of the lemma's assertion that $p(z') \cdot f' + V(a, z') = p(z') \cdot g'$ for some $g' \in \mathcal{P}_a(z')$. By upper continuity of $\succsim_a$, it follows that $F(a, z')$ is open relative to $X_a(z')$. Thus, there exists some $\lambda \in (0, 1)$ such that $\lambda f' \in X_a(z')$ and $(\lambda f', z') \succ_a (f(a), z)$. Hence, $\lambda p(z') \cdot f' + V(a, z') \geqslant I(a, z')$. Therefore, $I(a, z') \leqslant \lambda p(z') \cdot f' + V(a, z') < p(z') \cdot f' + V(a, z') = p(z') \cdot g' \leqslant I(a, z')$.
This is a contradiction, showing the assertion. ■

E.1 Proof of Theorem 4.14(a)

Suppose that for some agent $a \in A$ there is some collective good configuration $z' \in \mathcal{Z}$ and consumption bundle $f' \in X_a(z')$ such that $(f', z') \succ_a (f(a), z)$.
By Assumption 4.12 it follows that $f' \notin \partial\mathbb{R}^\ell_+$. Thus, $f' \gg 0$, implying that $p(z') \cdot f' > 0$.
Lemma E.2 immediately implies now that $p(z') \cdot f' + V(a, z') > p(z') \cdot g'$ for every $g' \in \mathcal{P}_a(z')$. This shows the assertion.

E.2 Proof of Theorem 4.14(b)

The following intermediary result shows that the premise of Lemma E.2 is always satisfied if the economy satisfies the hypotheses of Assumption 4.13.

Lemma E.3 *Under Assumption 4.13, for almost every agent $a \in A$ and every collective good configuration $z' \in \mathcal{Z}$ it holds that $t(a, z') > 0$.*

Proof. Let $z' \in \mathcal{Z}$. Define

$$T_2 = \{a \in A \mid t(a, z') = 0\} = \{a \in A \mid V(a, z') = I(a, z')\} \tag{21}$$

$$T_1 = A \setminus T_2 \tag{22}$$

Note that from Assumption 4.13(i) it follows that there exist $g\colon A \to \mathbb{R}^\ell$ and $\tilde{c} \in C(z')$ such that $\tilde{c} \ll \int g \, d\mu$. Hence, $\int V(\cdot, z') \, d\mu \leqslant p(z') \cdot \tilde{c} < \int I(\cdot, z') \, d\mu$. Thus, $\mu(T_1) > 0$.

We now show that $\mu(T_2) = 0$.
Suppose to the contrary that $\mu(T_2) > 0$. Then by Assumption 4.13(ii), there exist $c' \in C(z')$ and two integrable functions f' and g' such that $f'(a) \in X_a(z')$ and $(f'(a), z') \succ_a (f(a), z)$ for all $a \in T_1$ and $g'(a) \in \mathcal{P}_a(z')$ for all $a \in A$ such that

$$\int_{T_1} f' \, d\mu + c' \leqslant \int_{T_2} g' \, d\mu - \int_{T_2} f' \, d\mu. \tag{23}$$

Since for every $a \in T_1$, $(f'(a), z') \succ_a (f(a), z)$ and (f, g, c, z) is supported by (p, V), it holds that $p(z') \cdot f'(a) + V(a, z') \geqslant I(a, z')$. From $t(a, z') > 0$, by definition, $V(a, z') = I(a, z') - t(a, z') < I(a, z')$ and, therefore, from the above $p(z') \cdot f'(a) > 0$. Hence, from Lemma E.2 it follows that for all $a \in T_1$: $p(z') \cdot f'(a) + V(a, z') > I(a, z')$.

Furthermore, from (23) and the definition of T_2 we derive that

$$\begin{aligned}
\int_{T_1} I(\cdot, z')\, d\mu &< \int_{T_1} p(z') \cdot f'\, d\mu + \int_{T_1} V(\cdot, z')\, d\mu = \\
&= \int_{T_1} p(z') \cdot f'\, d\mu + \int V(\cdot, z')\, d\mu - \int_{T_2} V(\cdot, z')\, d\mu \leqslant \\
&\leqslant \int_{T_1} p(z') \cdot f'\, d\mu + p(z') \cdot c' - \int_{T_2} V(\cdot, z')\, d\mu \leqslant \\
&\leqslant \int_{T_2} p(z') \cdot g'\, d\mu - \int_{T_2} p(z') \cdot f'\, d\mu - \int_{T_2} V(\cdot, z')\, d\mu \leqslant \\
&\leqslant \int_{T_2} I(\cdot, z')\, d\mu - \int_{T_2} p(z') \cdot f'\, d\mu - \int_{T_2} I(\cdot, z')\, d\mu = \\
&= - \int_{T_2} p(z') \cdot f'\, d\mu \leqslant 0.
\end{aligned}$$

This is a contradiction, since for all $a \in A$: $I(a, z') \geqslant 0$.

Therefore, we conclude that $\mu(T_2) = 0$ and that $V(a, z') < I(a, z')$ or $t(a, z') > 0$ for almost every $a \in A$, showing the assertion. ■

Now let $(f', z') \succ_a (f(a), z)$. Since (f, g, c, z) is supported by (p, V), it follows that $p(z') \cdot f' + V(a, z') \geqslant I(a, z')$. Since from Lemma E.3 $t(a, z') > 0$, by definition $V(a, z') = I(a, z') - t(a, z') < I(a, z')$ and, therefore, from the above $p(z') \cdot f' > 0$.
Now Lemma E.2 immediately implies the assertion of Theorem 4.14(b).

F Proof of Theorem 5.2

We first establish a fundamental property of the attainment of maximal incomes if production sets are delimited.

Lemma F.1 *For every agent $a \in A$, every collective good configuration $z \in \mathcal{Z}$ and every price vector $p \in \mathbb{R}^\ell_+ \setminus \{0\}$, if $\mathcal{P}_a(z)$ is delimited, then $\sup p \cdot \mathcal{P}_a(z) = \max p \cdot \mathcal{P}_a(z)$.*

Proof. Since $\mathcal{P}_a(z)$ is delimited as assumed, there exists a compact set $\overline{\mathcal{P}}_a(z)$ such that $\mathcal{P}_a(z) = \overline{\mathcal{P}}_a(z) - \mathbb{R}^\ell_+$. Thus, $\sup p \cdot \mathcal{P}_a(z) \geqslant \max p \cdot \overline{\mathcal{P}}_a(z)$. Conversely, for any $g \in \mathcal{P}_a(z)$, there exists $\bar{g} \in \overline{\mathcal{P}}_a(z)$ and $x \in \mathbb{R}^\ell_+$ such that $g = \bar{g} - x$. Hence, $p \cdot g \leqslant p \cdot \bar{g} \leqslant \max p \cdot \overline{\mathcal{P}}_a(z)$ for all $g \in \mathcal{P}_a(z)$. Therefore, $\sup p \cdot \mathcal{P}_a(z) \leqslant \max p \cdot \overline{\mathcal{P}}_a(z)$, and hence $\sup p \cdot \mathcal{P}_a(z) = \max p \cdot \overline{\mathcal{P}}_a(z)$. ■

Next, we show that the price mechanism—as implemented here through the notion of supporting allocations using prices and valuations—introduces a dichotomy of production and consumption decisions at the level of the individual economic agent. This lemma extends the dichotomy stated in Gilles (2019b) for economies without collective goods. This dichotomy is crucial for our proof of Theorem 5.2.

Lemma F.2 (Dichotomy of consumption and production decisions)
Let the agent $a \in A$ be represented by $(X_a, \mathcal{P}_a, \succsim_a)$ such that the production set $\mathcal{P}_a(z)$ is delimited for every $z \in \mathcal{Z}$.
Let $z \in \mathcal{Z}$ be a given collective good configuration. Consider the following two-step optimisation problem for agent a:

Income maximisation: *The production plan $g^* \in \mathcal{P}_a(z)$ solves*

$$\max \{ p(z) \cdot g \mid g \in \mathcal{P}_a(z) \} . \tag{24}$$

Define $I(a,z) = p(z) \cdot g^* = \max p(z) \cdot \mathcal{P}_a(z)$.

Demand problem: *Given $g^* \in \mathcal{P}_a(z)$, the pair (f^*, z^*) maximises the preference relation $\succsim_a$ for agent $a \in A$ on the modified budget set*

$$\hat{B}(a,p,V) = \left\{ (x,z) \in \mathbb{R}^\ell_+ \times \mathcal{Z} \,\middle|\, p(z) \cdot x + V(a,z) \leqslant I(a,z) = p(z) \cdot g^* \right\} . \tag{25}$$

Then the following statements hold:

(a) *Let (f^*, g^*, z^*) solve the consumer-producer problem for a. If a is non-satiated at (f^*, z^*) and $\succsim_a$ satisfies Assumption 4.4, then (f^*, g^*, z^*) solves the two-step optimisation problem introduced above.*

(b) *Let (f^*, g^*, z^*) solve the two-step optimisation problem introduced above. Then (f^*, g^*, z^*) solves the consumer-producer problem for a.*

Proof. To prove the assertion (a) of the lemma, we first assume that the triple (f^*, g^*, z^*) optimises the preference relation $\succsim_a$ on the budget set $B(a,p,V)$ as defined in (8). The proof now proceeds in two steps:

We first show that g^* solves the income maximisation problem for z^*.
Indeed, assume that there exists some $g' \in \mathcal{P}_a(z^*)$ with $p(z^*) \cdot g' > p(z^*) \cdot g^*$.
Since by hypothesis agent a is non-satiated at (f^*, z^*), it follows from Assumption 4.4 and a similar argument as used in the proof of Proposition 4.5, it follows that $p(z^*) \cdot f^* + V(a,z^*) \leqslant p(z^*) \cdot g^* < p(z^*) \cdot g'$. Hence, there exists some $x \in X_a(z^*)$ with $(x, z^*) \succ_a (f^*, z^*)$ and $p(z^*) \cdot x + V(a,z^*) \leqslant p(z^*) \cdot g'$. Hence, $(x, g', z^*) \in B(a,p,V)$, which contradicts the maximality of (f^*, g^*, z^*) for $\succsim_a$.

Second, we show that the pair (f^*, z^*) is $\succsim_a$-maximal in $\hat{B}(a,p,V)$.
Suppose that there exists some $(f,z) \in \hat{B}(a,p,V)$ with $(f,z) \succ_a (f^*, z^*)$. Then, by the assertion that (f^*, g^*, z^*) is $\succsim_a$-maximal in the budget set $B(a,p,V)$, for any $g \in \mathcal{P}_a(z)$ it has to hold that $(f,g,c,z) \notin B(a,p,V)$. In particular, this has to hold for $\tilde{g} \in \mathcal{P}_a(z)$ with $p(z) \cdot \tilde{g} = \max\, p(z) \cdot \mathcal{P}_a(z)$ (Lemma F.1). Thus, $p(z) \cdot f + V(a,z) > p(z) \cdot \tilde{g} = \max\, p(z) \cdot \mathcal{P}_a(z) = I(a,z)$, which is a contradiction to $(f,z) \in \hat{B}(a,p,V)$.

To show assertion (b) of the lemma, suppose that the triple (f^*, g^*, z^*) solves two-stage optimisation problem given by the income and the demand problems as stated above. Now assume by contradiction that there exists another triple $(f,g,z) \in B(a,p,V)$ such that $(f,z) \succ_a (f^*, z^*)$. Then, since the pair (f^*, z^*) solves the demand problem and by assumption $(f,z) \notin \hat{B}(a,p,V)$, it follows that $p(z) \cdot f + V(a,z) > \max\, p(z) \cdot \mathcal{P}_a(z) \geqslant p(z) \cdot g$, which is a contradiction. ■

F.1 Proof of Theorem 5.2(a)

Consider an economy $\mathbb{E}$ as asserted in the theorem. Let (f^*, g^*, c^*, z^*) be a valuation equilibrium with price-valuation system (p, V).
Then, in particular, for every $a \in A$, the triple $(f^*(a), g^*(a), z^*)$ optimises $\succsim_a$ on $B(a,p,V)$ as defined in (8). From Lemmas F.1 and F.2 it follows that

$$p(z^*) \cdot g^*(a) = \max p(z^*) \cdot \mathcal{P}_a(z^*) = \sup p(z^*) \cdot \mathcal{P}_a(z^*).$$

Furthermore, by WIRSpec, there exists some set of ℓ full specialisation production plans $\mathcal{Q}_a(z^*)$ such that $\mathcal{Q}_a(z^*) \subset \mathcal{P}_a(z^*) \subset \operatorname{Conv} \mathcal{Q}_a(z^*) - \mathbb{R}^\ell_+$. Therefore,

$$\begin{aligned} \max\, p(z^*) \cdot \mathcal{Q}_a(z^*) \leqslant \max\, p(z^*) \cdot \mathcal{P}_a(z^*) \leqslant \\ \leqslant \sup\, p(z^*) \cdot \left(\operatorname{Conv} \mathcal{Q}_a(z^*) - \mathbb{R}^\ell_+ \right) = \\ = \sup\, p(z^*) \cdot \operatorname{Conv} \mathcal{Q}_a(z^*) = \max\, p(z^*) \cdot \mathcal{Q}_a(z^*). \end{aligned}$$

Hence,

$$p(z^*) \cdot g^*(a) = \sup\, p(z^*) \cdot \mathcal{P}_a(z^*) = \max\, p(z^*) \cdot \mathcal{Q}_a(z^*) \tag{26}$$

showing the assertion.

F.2 Proof of Theorem 5.2(b)

Additionally suppose SIRSpec. Hence, for every $a \in A$: $\mathcal{Q}_a(z^*) = \operatorname{Conv} \mathcal{Q}_a(z^*) \cap \mathcal{P}_a(z^*)$.
Now, from (26), it follows that $p(z^*) \cdot g^*(a) = \max\, p(z^*) \cdot \mathcal{Q}_a(z^*)$.
Suppose that $g^*(a) \in \mathcal{P}_a(z^*) \setminus \mathcal{Q}_a(z^*)$, then $g^*(a) \notin \operatorname{Conv} \mathcal{Q}_a(z^*)$ implying that there exist some $y \in \operatorname{Conv} \mathcal{Q}_a(z^*)$ and some $k \in \{1, \ldots, \ell\}$ with $g^{*k}(a) < y^k$. Therefore, since $p(z^*) \gg 0$,

$$p(z^*) \cdot g^*(a) < p(z^*) \cdot y \leqslant \max p(z^*) \cdot \operatorname{Conv} \mathcal{Q}_a(z^*) = \max\, p(z^*) \cdot \mathcal{Q}_a(z^*),$$

which is a contradiction.

From the fact that $g^*(a) \in \mathcal{Q}_a(z^*)$, we may introduce for every $k \in \{1, \ldots, \ell\}$: $A_k(g^*) = \{a \in A \mid g^*_k(a) > 0\} \in \Sigma$. It is obvious that the collection $\{A_1(g^*), \ldots, A_\ell(g^*)\}$ forms partitioning of the agent set A such that $\sum_{k=1}^{\ell} \mu(A_k(g^*)) = \mu(A) = 1$.

F.3 Proof of Theorem 5.2(c)

Now with reference to Lemma F.2, it follows that in equilibrium every agent $a \in A$ maximises her income under price system $p(z^*)$. Hence, for every $a \in A$: $p(z^*) \cdot g^*(a) = \max\, p(z^*) \cdot \mathcal{P}_a(z^*)$.
Furthermore, by the uniform specialisation property of $\mathbb{E}$, there exists a common set of full specialisation production plans $\mathcal{Q}(z^*) = \left\{ \hat{y}^k(z^*) \,\middle|\, k = 1, \ldots, \ell \right\}$ for which the SIRSpec property holds for every $a \in A$. Hence, for all $k \in \{1, \ldots, \ell\}$ it follows that for every $a \in A_k(g^*)$: $p(z^*) \cdot g^*(a) = p(z^*) \cdot \hat{y}^k(z^*) = \max\, p(z^*) \cdot \mathcal{Q}(z^*)$.
Since $\int g^*\, d\mu \gg 0$ it follows that $\mu(A_k(g^*)) > 0$ for every $k \in \{1, \ldots, \ell\}$. Hence, $p(z^*) \cdot \hat{y}^k(z^*) = p(z^*) \cdot \hat{y}^m(z^*) = \max\, p(z^*) \cdot \mathcal{Q}(z^*)$ for all commodities $k \neq m$.
This completes the proof of the assertion.