Probabilistic Network Values*

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Abstract

We consider a class of cooperative network games with transferable utilities in which players interact through a probabilistic network rather than a regular, deterministic network. In this class of wealth-generating situations we consider probabilistic extensions of the Myerson value and the position value. For the subclass of probabilistic network games in multilinear form, we establish characterizations of these values using an appropriate formulation of component balancedness. We show axiomatizations based on extensions of the well-accepted properties of equal bargaining power, balanced contributions, and balanced link contributions.

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1 Value theory and networks

Considering the effects of the introduction of communication networks into cooperative game theory dates back to Myerson [13]. He introduced a perspective that networks represent communication structures that constrain interactions between players. Therefore, coalitions are only formable or "feasible" if they are connected in the communication network. These configurations of a communication network and a cooperative game with transferable utilities are known as *communication situations*. Myerson considered the restrictions of standard cooperative TU-games on the class of these feasible coalitions and investigated the Shapley value of these restricted games—subsequently known as the *Myerson value* on the class of communication situations. Myerson showed that this

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value is completely characterized by the two properties of *component balance* and *equal bargaining power* on the subclass of component efficient communication situations.

A dualistic perspective on value allocation in communication situations was originally formulated by Meessen [11] and subsequently developed by Borm et al. [4]. In this approach, communication links rather than players are considered as the source of all generated wealth. This transforms the communication situation into a *link game* in which communication links act as players. The Shapley value of the link game now assigns fair values to all links in the communication network based on the generated wealth . The *position value* of a communication situation is now the equal assignment of these Shapley link values to all constituting players of these links. The position value is completely characterized by *component balance* and the *balanced link contribution property* on the subclass of component efficient communication situations—as shown by Slikker [17].

In communication situations, the architecture of the communication network is not important. Two different networks inducing the same partition of the player set will yield an identical restricted game and, hence, identical Myerson and position values. Jackson and Wolinsky [8] introduced games in which value stems directly from the network rather than a coalition of players. This is referred to as a *network game*.

The extension introduced by Jackson and Wolinsky represents a different perspective on how collective wealth is created. In a network game, the network facilitates the creation of value, rather than constrains it. So, unlike for communication situations, in a network game all value emanates from the facilitated interaction between players in the network. In this perspective, the formation or deletion of links affects wealth generation directly and, therefore, is rather consequential for the analysis of value allocation in these network games.

Allocation rule extensions for network games were pursued by Jackson and Wolinsky [8] for the Myerson value and by Slikker [18] for the position value. Both extensions can be characterized for component efficient network games by straightforward extensions of the same fundamental axioms as used in the axiomatizations developed for component efficient communication situations.

Probabilistic extensions. The notion of a probabilistic network has been introduced by Calvo et al. [5] in the context of communication situations. In a probabilistic network, every link's formation is probabilistic and forms with a given link formation probability[1]. All link formation probabilities are assumed to be independent, which allows for a natural determination of the probability with which a network actualises. The value of any coalition is then the expected value of the fixed value based on the restricted game created for all possible randomly created networks on that coalition. This framework is referred to as a *probabilistic communication situation*.

Calvo et al. [5] considered extensions of the Myerson and position values to the class of probabilistic communication situations. They used probabilistic extensions of the familiar axiomatic properties of component balancedness, and equal bargaining power[2] to characterize the extension of the Myerson value to the class of probabilistic communication situations.

[1]Probabilistic network Games have also been studied in the context of non-cooperative Games (see, e.g., Billand et al. [1] and the references therein.)

[2]The later property is denoted as "fairness" in Calvo et al. [5], following Myerson [13]'s terminology adopted in his seminal paper on communication situations.

Gomez et al. [7] consider a further generalization of the probabilistic framework considered in Calvo et al. [5]. They dispense with the independence assumption and consider any arbitrary probability distribution on the set of all possible networks involving all players. Hence, the probability of the formation of a certain network is no longer assumed to be based on independent link formation probabilities. This is referred to as a *generalized probabilistic communication situation.* They extend the Myerson value and characterize it on the class of component efficient generalized probabilistic communication situations using appropriate extensions of the component balance and equal bargaining power properties. Ghintran et al. [6] define and similarly characterize an extension of the position value to such generalized probabilistic communication situations.

Extension to probabilistic network games. In this paper we extend Calvo et al. [5]'s conception of a probabilistic network to the realm of network games. In a communication situation, the network specifying the communication restrictions is given. Calvo et al. assigned a probability distribution over the set of all possible networks. Here, we extend this reasoning to network games in which formed links facilitate wealth creation rather than constrain it. Hence, even though links are formed probabilistically, the generated network as a whole determines the wealth that is generated. This allows the incorporation of widespread externalities from the network architecture. We refer to this class of wealth generating frameworks as *probabilistic network games.*

Throughout, we assume—à la Calvo et al. [5]—that each link between two arbitrary players is formed according to a pre-specified link probability. A probabilistic network game now assigns to each configuration of link probabilities a value. This implies that we take full account of the different probabilistic configurations on all potential links. This captures widespread externalities that different probabilities of a link forming affect the overall wealth generated in the resulting situation.

A practical example of probabilistic network games is airline code sharing networks. Passengers travelling on intercontinental flights often use multiple airlines who have code sharing agreements with one another. The passenger pays an up-front fee which is divided among the relevant code sharing airlines according to a given allocation rule. But given the competition among airlines, these links are often unstable with airlines terminating existing agreements and forming new agreements. This probabilistic reconfiguration affects overall performance of the network.

Throughout we limit ourselves to probabilistic network games that are *component efficient.* This requirement imposes that externalities only extend to components of the probabilistic network and that there are no widespread externalities across such components. This seems a natural restriction, since wealth is created through links between players as an omission of such links prevents such wealth to be generated.

Probabilistic values and their characterizations. Within our extended framework of probabilistic network games, we consider appropriate extensions of the Myerson and position values—denoted as *probabilistic* Myerson and position values. These probabilistic values are introduced as the expected allocated values over all possible network games that can emerge from a probabilistic network game. This straightforwardly follows and extends the approach to defining probabilistic values followed by Calvo et al. [5] as well as Gomez et al. [7].

To properly develop a characterization of the probabilistic Myerson and position values, we restrict ourselves to the subclass of so-called *multilinear* probabilistic network games which take a multilinear form. In particular, multilinear probabilistic network games are founded on the use of Owen's [14, 15] conception of the multilinear extension of a cooperative game. In Owen's seminal formulation, each player joins a coalition with a given probability. Now, the multilinear extension is the resulting expectation of the collective wealth generated by that particular probabilistic coalition. We use a similar formulation to derive a probabilistic network game as the multilinear form of the underlying network game, given the probabilities that links between players are formed in the probabilistic network.

On the subclass of multilinear probabilistic network games we are able to fully develop and implement characterizations of the probabilistic Myerson and position values, extending the insights developed in Calvo et al. [5] as well as Slikker [18]. In particular, we show that (1) the probabilistic Myerson value is the unique probabilistic network allocation rule satisfying component balance as well as equal bargaining power; (2) the probabilistic Myerson value is the unique probabilistic network allocation rule satisfying component balance as well as the balanced contributions property; and (3) the probabilistic position value is the unique probabilistic network allocation rule satisfying component balance as well as the balanced link contributions property.

Outline of the paper: The rest of the paper is as follows. In Section 2, we discuss the notation and terminology related to (deterministic) network games. Subsequently, we introduce the appropriate notion of a probabilistic network and a probabilistic network game. Finally, we introduce the subclass of multilinear probabilistic network games, using a multilinear extension formulation. Section 3 considers the probabilistic Myerson value as an appropriate extension of the definition introduced by Calvo et al. [5] to the class of multilinear probabilistic network games. We also show the two main characterizations of this probabilistic Myerson value. Section 4 introduces our notion of the probabilistic position value to the class of multilinear probabilistic network games. We establish the main characterization of this probabilistic position value founded on component balancedness and the balanced link contributions property. Section 5 concludes.

2 Probabilistic Networks and Games

Throughout this paper, we let $N = \{1, 2, \ldots, n\}$ be a finite, fixed set of players. Given the player set N, a **link** represents an undirected relationship between two players. Formally, a link between players i and j is defined as $ij = \{i, j\}$. Clearly, ij is equivalent to ji.

The collection of all possible links is denoted as $g_N = \{ij \mid i, j \in N \text{ and } i \neq j\}$. A **network** is defined as any collection of links, i.e., $g \subseteq g_N$. Clearly, g_N itself can also be referred to as the *complete network* on N

The set of all possible networks on N is $\mathbb{G}^N = \{g \mid g \subseteq g_N\}$. The network $g_0 = \varnothing$ is the network without any links, referred to as the *empty network*. Let the number of links in a network g be denoted by $\#g$. Obviously, $\#g_N = \binom{n}{2} = \frac{1}{2}n(n-1)$ and $\#g_0 = 0$.

For every network $g \in \mathbb{G}^N$ and every player $i \in N$ we denote i's *neighborhood* in g by $N_i(g) = \{j \in N \mid j \neq i \text{ and } ij \in g\}$ as the set of players with whom i is directed linked in g. Also, denote $n_i(g) = \#N_i(g)$. Alternatively, i's neighborhood can be represented by her *link set* $L_i(g) = \{ij \in g \mid j \in N_i(g)\} \subseteq g$.

We also define $N(g) = \cup_{i \in N} N_i(g) \subseteq N$ as the set of connected players in g and let $n(g) = \#N(g)$ with the convention that if $N(g) = \emptyset$, we let $n(g) = 1$.[3] Furthermore, $n(g)$ will be referred to as *size* of the network g.

A *path* in g connecting i and j is a set of distinct players $\{i_1, i_2, \ldots, i_p\} \subseteq N(g)$ with $p \geqslant 2$ such that $i_1 = i$, $i_p = j$, and $\{i_1 i_2, i_2 i_3, \ldots, i_{p-1} i_p\} \subseteq g$. We say i and j are *connected* to each other if a path exists between them and they are *disconnected* otherwise.

The network $g' \subseteq g$ is a **component** of g if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies $ij \in g'$. In other words, a component is simply a maximally connected subnetwork of g. We denote the set of network components of the network g by $C(g)$.

The set of players that are not connected in the network g are collected in the set of *isolated players* in g denoted by $N_0(g) = N \setminus N(g) = \{i \in N \mid N_i(g) = \emptyset\}$. Clearly, $N_0(g_0) = N$.

Furthermore, for any $g \in \mathbb{G}^N$ and $S \subseteq N$, we introduce a subnetwork $g|_S$ as the restriction of g on the player set S. Formally, we let $g|_S = g \cap g_S = \{ij \in g \mid i, j \in S\}$ where $g_S = \{ij \in g_N \mid i, j \in S \subseteq N \text{ and } i \neq j\}$.

Finally, for $g \subseteq g_N$ and $g' \subseteq g_N \setminus g$, let $g + g'$ denote the network $g \cup g'$. Similarly, for $g' \subseteq g$, let $g - g'$ denote the network $g \setminus g'$.

2.1 Network games and allocation rules

Under the framework investigating the allocation of transferable utility in a network pioneered by Jackson and Wolinsky [8], we introduce our main framework.[4]

Definition 2.1 *A* ***network game*** *is defined as a function* $w \colon \mathbb{G}^N \to \mathbb{R}$ *such that the following two properties are satisfied:*

(i) $w(g_0) = 0$*, and*

(ii) **Component additivity:** *For every network* $g \in \mathbb{G}^N$*:*

$$w(g) = \sum_{h \in C(g)} w(h) \tag{2.1}$$

The space of all network games is denoted by $\mathbb{H}^N$*, which is a finite dimensional Euclidean vector space.*

Component additivity imposes that there are no widespread externalities across different components in a network. Hence, wealth is cooperatively created strictly between and among connected

[3] We emphasize here that if $N(g) \neq \emptyset$, we have that $n(g) \geqslant 2$. Namely, in those cases the network has to consist of at least one link.

[4] Throughout this paper, we follow the suggested notational and naming convention in Jackson [9] and Borkotokey et al.[3].

players only. Isolated players do not generate any wealth and do affect the wealth created by other players either.

Definition 2.2 *A **network allocation rule** is a function $Y\colon \mathbb{G}^N \times \mathbb{H}^N \longrightarrow \mathbb{R}^N$ such that for every $g \in \mathbb{G}^N$ and every $w \in \mathbb{H}^N$ it holds that $\sum_{i \in N} Y_i(g,w) = w(g)$ and $Y_k(g,w) = 0$ for all isolated nodes $k \in N_0(g)$.*

It follows from Definition 2.2 that a network allocation rule determines how the collective value generated within a certain given network $g \in \mathbb{G}^N$ under network game $w \in \mathbb{H}^N$ is distributed over the individual players $i \in N$. In particular, $Y_i(g,w)$ is the payoff to player i in the network g under the network game w. Since values are generated explicitly through links, it is logical that non-participating isolated nodes are assigned zero allocations accordingly.

In this paper we investigate extensions of two prominent network allocation rules. Myerson [13] seminally investigated the Shapley value for component additive communication situations. Jackson and Wolinsky [8] extended this allocation rule to the setting of network games.

The second prominent allocation rule is based on a dualistic approach to the allocation of generated wealth in a communication network, namely through the assignment of an allocated value to each link rather than to each player, which are subsequently fairly divided among the two constituting players on each link. This was developed by Borm et al. [4] and Slikker [17] for communication situations and extended by Slikker [18] to network games.

Definition 2.3 *We define the following two allocation rules on $\mathbb{G}^N \times \mathbb{H}^N$:*

(a) *The **Myerson allocation rule** is the network allocation rule $Y^m\colon \mathbb{G}^N \times \mathbb{H}^N \longrightarrow \mathbb{R}^N$ given by*

$$Y_i^m(g,w) = \sum_{S \subseteq N \setminus \{i\}} \left(w(g|_{S \cup i}) - w(g|_S) \right) \left(\frac{|S|!(n-|S|-1)!}{n!} \right). \tag{2.2}$$

for every $g \in \mathbb{G}^N$ and $w \in \mathbb{H}^N$.

(b) *The **position allocation rule** is the network allocation rule $Y^p\colon \mathbb{G}^N \times \mathbb{H}^N \longrightarrow \mathbb{R}^N$ given by*

$$Y_i^p(g,w) = \tfrac{1}{2} \sum_{ij \in g} \sum_{g' \subseteq g \setminus ij} \frac{(\#g'!)\,(\#g - \#g' - 1)!)}{(\#g!)} \left(w(g' + ij) - w(g') \right) \tag{2.3}$$

for every $g \in \mathbb{G}^N$ and $w \in \mathbb{H}^N$.

We investigate these two allocation rules in the context of the following additional properties.

Definition 2.4 *Let $Y\colon \mathbb{G}^N \times \mathbb{H}^N \longrightarrow \mathbb{R}^N$ be some network allocation rule on player set N.*

(a) *Allocation rule Y is **component balanced** if for every $w \in \mathbb{H}^N$ and every $g \in \mathbb{G}^N$ it holds that $\sum_{i \in N(h)} Y_i(g,w) = w(h)$ for every component $h \in C(g)$.*

(b) *Allocation rule Y satisfies **equal bargaining power** if for every network $g \in \mathbb{G}^N$, every value function $w \in \mathbb{H}^N$, and all players $i,j \in N$ such that $ij \in g$:*

$$Y_i(g,w) - Y_i(g - ij, w) = Y_j(g,w) - Y_j(g - ij, w) \tag{2.4}$$

(c) *Allocation rule Y satisfies the **balanced contributions property** if for every network $g \in \mathbb{G}^N$, every value function $w \in \mathbb{H}^N$, and for all players $i, j \in N$*

$$Y_i(g, w) - Y_i\left(g - L_j(g), w\right) = Y_j(g, w) - Y_j\left(g - L_i(g), w\right) \tag{2.5}$$

(d) *Allocation rule Y satisfies the **balanced link contributions property** if for every network $g \in \mathbb{G}^N$, every value function $w \in \mathbb{H}^N$, and for all players $i, j \in N$*

$$\sum_{jk \in L_j(g)} \left(Y_i(g, w) - Y_i(g - jk, w)\right) = \sum_{ik \in L_i(g)} \left(Y_j(g, w) - Y_j(g - ik, w)\right) \tag{2.6}$$

The properties stated above have been used to characterize the Myerson allocation rule as well as the position allocation rule introduced above. Jackson and Wolinsky [8] showed that the Myerson allocation rule is the unique network allocation rule satisfying both component balance and equal bargaining power. Slikker [18] showed that the Myerson allocation rule is the unique network allocation rule satisfying component balance and the balanced contributions property.

Slikker [18] also showed that the position allocation rule is the unique network allocation rule satisfying component balance as well as the balanced link contributions property.

These properties are collected in the following proposition:

Proposition 2.5 *Let $Y : \mathbb{G}^N \times \mathbb{H}^N \to \mathbb{R}^N$ be some network allocation rule. Then the following properties hold:*

(i) Jackson and Wolinsky [8]: *$Y = Y^m$ if and only if Y is component balanced and Y satisfies the equal bargaining power property.*

(ii) Slikker [18]: *$Y = Y^m$ if and only if Y is component balanced and Y satisfies the balanced contributions property.*

(iii) Slikker [18]: *$Y = Y^p$ if and only if Y is component balanced and Y satisfies the balanced link contributions property.*

2.2 Probabilistic network games

In the realm of network games, the interaction between players is considered to be fully deterministic. One either has a relationship or not. Here we turn to the more general framework in which relationships are probabilistic. Hence, two players interact through a relationship that might break down or fail with a certain probability. This framework introduces essentially a transaction cost perspective on network formation in which links might fail to materialise under high cost stresses.

Probabilistic networks: Formally, a *probabilistic network* on player set $N = \{1, \dots, n\}$ is a function $p : g_N \to [0, 1]$ that assigns to every link $ij \in g_N$ a probability $p_{ij} \in [0, 1]$ that the link is forming. These probabilities are assumed to be independent. Hence, the probability of network

$g \in \mathbb{G}^N$ forming is given by

$$\mu(g, p) = \prod_{ij \in g} p_{ij} \times \prod_{ij \notin g} (1 - p_{ij}). \tag{2.7}$$

In this fashion the probabilistic network p generates a probability distribution $\mu(\cdot, p)$ on the set of all possible (regular) networks $\mathbb{G}^N$. When there is no ambiguity of the player set N, we denote the probabilistic network simply by the probability function p. We introduce $\mathbb{P}^N = \{p : g_N \to [0,1]\} = [0,1]^{\binom{n}{2}}$ as the set of all probabilistic networks on the player set N.

There are two extreme probabilistic networks: $p_0 \in \mathbb{P}^N$ defined by $p_0(ij) = 0$ for all $ij \in g_N$ represents the *empty* probabilistic network in which no links are formed; and $p_N \in \mathbb{P}^N$ defined by $p_N(ij) = 1$ for all $ij \in g_N$ is the *complete* probabilistic network in which all links are formed with complete certainty.

For a probabilistic network $p \in \mathbb{P}^N$, the **support** of p—denoted by $g(p) \subseteq g_N$—is the network consisting of links that form with positive probabilities: $g(p) = \{ij \in g_N \mid p_{ij} > 0\} \in \mathbb{G}^N$. Clearly, $g(p_0) = g_0 = \varnothing$ and $g(p_N) = g_N$.

It is easy to see that the set of all networks $\mathbb{G}^N$ can be conceived as a subclass of the set of probabilistic networks $\mathbb{P}^N$. Indeed, for each $g \in \mathbb{G}^N$, we can define a function $e^g : g_N \to \mathbb{R}$ by $e^g(ij) = e^g_{ij} = 1$, if $ij \in g$, and $e^g(ij) = e^g_{ij} = 0$, otherwise. Clearly $e^g \in \mathbb{P}^N$ is a probabilistic network descriptor of the (deterministic) network $g \in \mathbb{G}^N$. Thus, with some abuse of notation we have $\mathbb{G}^N \subseteq \mathbb{P}^N$.

A **subnetwork** of a probabilistic network $p \in \mathbb{P}^N$ is any probabilistic network $p' \in \mathbb{P}^N$ such that $p'_{ij} \in \{0, p_{ij}\}$ for all $ij \in g_N$. Therefore, each probabilistic network p induces $2^{\#g(p)}$ probabilistic subnetworks. For a regular network $h \in \mathbb{G}^N$ we define the corresponding (probabilistic) subnetwork as $p^h \in \mathbb{P}^N$ defined by $p^h_{ij} = \min\{p_{ij}, e^h_{ij}\}$. Note that for each $h \subseteq g(p)$: $g(p^h) = h$.

Two players i and j are **connected** in the probabilistic network $p \in \mathbb{P}^N$ if i and j are connected in its support $g(p)$. Hence, i and j are connected if with a positive probability there exists a path between them in a realisation $h \subseteq g(p)$ of the probabilistic network p.

A *component* in a probabilistic network p is a maximally connected subset of N. Hence, the components of p are simply the components $C(g(p))$ of its support $g(p)$. For each component $h \in C(g(p))$, the probabilistic subnetwork $p^h \in \mathbb{P}^N$ is a *probabilistic component* of p.

Probabilistic network games and allocation rules. We generalise the concept of a network game from the domain of regular, deterministic network to the extended domain of probabilistic networks.

Definition 2.6 *A* ***probabilistic network game*** *is a function $v : \mathbb{P}^N \to \mathbb{R}$ such that the following two properties hold:*

(i) $v(p_0) = 0$, *and*

(ii) **Component additivity:** *for all* $p \in \mathbb{P}^N$:

$$v(p) = \sum_{h \in C(g(p))} v\left(p^h\right). \tag{2.8}$$

The space of all probabilistic network games is denoted by $\mathbb{V}^N$.

As before, component additivity imposes the conception that wealth is only created among connected players. Isolated players that have zero probability is being connected to any other player are, therefore, assumed to not participate in any wealth creation. Widespread externalities are excluded beyond components in the probabilistic network.

We can now introduce the main object of our analysis, namely the allocation of generated wealth in a probabilistic network game.

Definition 2.7 *A* ***probabilistic network allocation rule*** *is a function* $\Psi\colon \mathbb{V}^N \times \mathbb{P}^N \to \mathbb{R}^N$ *such that for every probabilistic network* $p \in \mathbb{P}^N$ *and every probabilistic network game* $v \in \mathbb{V}^N$ *it holds that* $\Psi_k(v, p) = 0$ *for all isolated nodes* $k \in N_0(g(p))$ *in the support of* p.

A probabilistic network allocation rule allocates the collective wealth generated in a probabilistic network game within a given probabilistic network among all its constituting players. Again, we explicitly assume that non-participating players, represented by isolated nodes in the support of the probabilistic network, are allocated a zero payoff.

Definition 2.8 *The probabilistic network allocation rule* $\Psi\colon \mathbb{V}^N \times \mathbb{P}^N \to \mathbb{R}^N$ *is* ***component balanced*** *if for every probabilistic network game* $v \in \mathbb{V}^N$ *and every probabilistic network* $p \in \mathbb{P}^N$:

$$\sum_{i \in N(h)} \Psi_i(v, p) = v(p^h) \tag{2.9}$$

for all components $h \in C(g(p))$ *in the support of* p.

This definition is valid due to the assumed component additivity of each probabilistic network game, implying in particular that the allocation rule is balanced in the sense that $\sum_{i \in N} \Psi_i(v, p) = v(p)$ for every probabilistic network game $v \in \mathbb{V}^N$ and every probabilistic network $p \in \mathbb{P}^N$.

2.3 Multilinear probabilistic network games

Our analysis will focus on a specific subclass of probabilistic network games that are closely associated with regular network games.

Definition 2.9 *Let* $w \in \mathbb{H}^N$ *be a network game. Then we define* $v_w\colon \mathbb{P}^N \to \mathbb{R}$ *where for every probabilistic network* $p \in \mathbb{P}^N$:

$$v_w(p) = \sum_{g \subseteq g(p)} w(g) \cdot \mu(g, p) \quad \text{where} \quad \mu(g, p) = \prod_{ij \in g} p_{ij} \times \prod_{ij \notin g} (1 - p_{ij}). \tag{2.10}$$

We first show the following property, which allows us to show that the defined framework is consistent and defines an appropriate subclass of probabilistic network games. Furthermore, the next property allows us to extend the Myerson and position allocation rules from the realm of network games into the setting of probabilistic network games.

Lemma 2.10 *For every network game $w \in \mathbb{H}^N$ consider the constructed mapping $v_w : \mathbb{P}^N \to \mathbb{R}$. Then it holds that for every network $h \in \mathbb{G}^N$ and every probabilistic network $p \in \mathbb{P}^N$:*

$$v_w(p^h) = \sum_{g \subseteq g(p)} \mu(g,p) \cdot w(g \cap h) \tag{2.11}$$

Proof. For multilinear forms we state the following well-known property that, for every $m \geqslant 2$, any finite set $M = \{1, 2, \cdots, m\}$ and numbers $\{x_1, \cdots, x_m\} \subset [0, 1]$ it holds that

$$\sum_{T \subseteq M} \left[\prod_{i \in T} x_i \prod_{i \notin T} (1 - x_i) \right] = 1. \tag{2.12}$$

Let $w \in \mathbb{H}^N$. Furthermore, let $p \in \mathbb{P}^N$ and $h \subseteq g(p)$ be given. Clearly, it holds that $g(p^h) = h$. Now,

$$\begin{aligned}
\sum_{g \subseteq g(p)} w(g \cap h)\, \mu(g,p) &= \sum_{g \subseteq g(p)} w(g \cap h) \prod_{ij \in g} p_{ij} \prod_{ij \notin g} (1 - p_{ij}) \\
&= \sum_{g \subseteq g(p)} w(g \cap h) \left\{ \prod_{ij \in g \cap h} p_{ij} \prod_{ij \in h \setminus g \cap h} (1 - p_{ij}) \prod_{ij \in g \setminus g \cap h} p_{ij} \prod_{ij \in g(p) \setminus g \cup h} (1 - p_{ij}) \right\} \\
&= \sum_{g \subseteq g(p)} w(g \cap h) \left\{ \prod_{ij \in g \cap h} p_{ij} \prod_{ij \in g(p^h) \setminus g \cap h} (1 - p_{ij}) \prod_{ij \in g \setminus g \cap h} p_{ij} \prod_{ij \in g(p) \setminus g \cup h} (1 - p_{ij}) \right\} \\
&= \sum_{g \subseteq g(p^h)} w(g \cap h) \cdot \mu(g \cap h, p^h) \left[\sum_{g' \subseteq g(p) \setminus h} \left\{ \prod_{ij \in g'} p_{ij} \prod_{ij \in g(p) \setminus g' \cup h} (1 - p_{ij}) \right\} \right] \\
&= \sum_{g \subseteq g(p^h)} w(g) \cdot \mu(g, p^h) \cdot 1 = \sum_{g \subseteq g(p^h)} w(g) \cdot \mu(g, p^h)
\end{aligned}$$

where the second to last step from below is valid due to (2.12).
Hence, we have shown that for the mapping v_w defined in (2.10) it holds that

$$v_w(p^h) = \sum_{g \subseteq g(p^h)} w(g) \cdot \mu(g, p^h) = \sum_{g \subseteq g(p)} w(g \cap h) \cdot \mu(g, p)$$

This completes the proof of the assertion. ■

The following lemma links the constructed extension of a network game to that of a probabilistic network game.

Lemma 2.11 *For every network game $w \in \mathbb{H}^N$, it holds that $v_w \in \mathbb{V}^N$.*

Proof. Let $w \in \mathbb{H}^N$ and by definition of v_w, it now holds for the empty probabilistic network p_0 that

$$v_w(p_0) = \sum_{g \subseteq g(p_0)} w(g) \cdot \mu(g, p_0) = w(g_0) \cdot \mu(g_0, p_0) = 0.$$

Let now $p \neq p_0$, $p \in \mathbb{P}^N$ be fixed. From the component additivity of w we derive that

$$\sum_{h \in C(g(p))} v_w\left(p^h\right) = \sum_{h \in C(g(p))} \left[\sum_{g \subseteq g(p^h)} w(g) \cdot \mu(g, p^h) \right] = \sum_{h \in C(g(p))} \left[\sum_{g \subseteq g(p)} w(g \cap h) \cdot \mu(g, p) \right]$$

$$= \sum_{g \subseteq g(p)} \sum_{h \in C(g(p))} w(g \cap h) \cdot \mu(g, p) = \sum_{g \subseteq g(p)} \mu(g, p) \cdot \left[\sum_{h \in C(g(p))} w(g \cap h) \right]$$

$$= \sum_{g \subseteq g(p)} \mu(g, p) \cdot \left[\sum_{h' \in C(g)} w(h') \right] = \sum_{g \subseteq g(p)} \mu(g, p) \cdot w(g) = v_w(p)$$

where we use Lemma 2.10 in the second step in the derivation above. This shows that v_w indeed is component additive and, therefore, $v_w \in \mathbb{V}^N$. ■

Lemma 2.11 shows that the constructed function associated to a network game is actually a probabilistic network game. This construction refers and compares to the multilinear extension of a standard cooperative game seminally developed by Owen [15]. The assertion of Lemma 2.11 allows us to introduce the following formalisation of this multilinear extension.

Definition 2.12 *For every network game $w \in \mathbb{H}^N$, the associated probabilistic network game $v_w \in \mathbb{V}^N$ is called the* ***multilinear extension*** *of w.*
A probabilistic network game $v \in \mathbb{V}^N$ is called a ***multilinear probabilistic network game*** *if there exists a network game $w \in \mathbb{H}^N$ such that $v = v_w$. For a multilinear probabilistic network game v, the corresponding network game w such that $v = v_w$ is also denoted as an* ***associated*** *network game.*
The subclass of all multilinear probabilistic network games is denoted by $\mathbb{W}^N = \{v_w \mid w \in \mathbb{H}^N\} \subset \mathbb{V}^N$.

Note that in the assertion of Lemma 2.10, when in particular $h = g_N$, we arrive at $p^h = p^{g_N} = p$ for each $p \in \mathbb{P}^N$ and, therefore, Lemma 2.10 reverts back to (2.10). This shows the consistency of the definitions underpinning the subclass $\mathbb{W}^N$ of multilinear probabilistic network games.

Multilinearity and allocation rules. Given a network allocation rule $Y : \mathbb{G}^N \times \mathbb{H}^N \to \mathbb{R}^N$, we define the *associated* probabilistic network allocation rule $\Psi_Y : \mathbb{W}^N \times \mathbb{P}^N \to \mathbb{R}^N$ on the class $\mathbb{W}^N$ of multilinear probabilistic network games as

$$\Psi_{Y,i}(v_w, p) = \sum_{g \subseteq g(p)} \left[Y_i(g, w) \cdot \mu(g, p) \right] \tag{2.13}$$

where $w \in \mathbb{H}^N$ is the associated network game of $v_w \in \mathbb{W}^N$.

Next we show that on the class of multilinear probabilistic network games, the associated probabilistic network allocation rule of any component balanced allocation rule is component balanced as well.

Proposition 2.13 *For any component balanced network allocation rule* $Y\colon \mathbb{G}^N \times \mathbb{H}^N \to \mathbb{R}^N$*, the associated probabilistic network allocation rule* $\Psi_Y\colon \mathbb{W}^N \times \mathbb{P}^N \to \mathbb{R}^N$ *is component balanced on the class of multilinear probabilistic network games* $\mathbb{W}^N \subset \mathbb{V}^N$*.*

Proof. Consider a multilinear probabilistic network game $v_w \in \mathbb{W}^N \subset \mathbb{V}^N$ with associated network game $w \in \mathbb{H}^N$. Now, clearly, both w and v_w are component additive by assumption.
Let $p \in \mathbb{P}^N$ be some probabilistic network. Take any component $h \in C(g(p))$ and any arbitrary subnetwork $g \subseteq g(p)$. In this subnetwork, there are no links, by definition, between any member $i \in N(h)$ and $j \in N \backslash N(h)$.
So, with regard to the players in $N(h)$, their corresponding subnetwork in g—given by $g \cap h$—is effectively a subset of h. Therefore, by the component additivity of w and component balance property of the allocation rule Y, it follows that $\sum_{i \in N(h)} Y_i(g,w) = w(g \cap h)$. Hence, by Lemma 2.10 it follows that

$$\sum_{i \in N(h)} \Psi_{Y,i}(v_w, p) = \sum_{i \in N(h)} \sum_{g \subseteq g(p)} Y_i(g,w) \cdot \mu(g,p) = \sum_{g \subseteq g(p)} \sum_{i \in N(h)} Y_i(g,w) \cdot \mu(g,p)$$

$$= \sum_{g \subseteq g(p)} \mu(g,p) \left[\sum_{i \in N(h)} Y_i(g,w) \right] = \sum_{g \subseteq g(p)} \mu(g,p) \cdot w(g \cap h) = v_w \left(p^h \right).$$

Finally, if $i \in N$ is an isolated node in p in the sense that $p_{ij} = 0$ for every $j \neq i$, then for every network $g \subseteq g(p)$ it has to hold that $N_i(g) = \varnothing$. This, in turn, implies that $Y_i(g,w) = 0$ and, hence, $\Psi_{Y,i}(v_w, p) = 0$.
This completes the proof. ■

Throughout the next sections of the paper, we focus mainly on extensions of the Myerson and position allocation rules to our setting. Component additivity as well as the component balance property of these particular allocation rules implies that the class of multilinear probabilistic network games $\mathbb{W}^N \subset \mathbb{V}^N$ seems the proper universe on which to consider these extensions.

3 The Probabilistic Myerson Value

We investigate the probabilistic network allocation rule based on the Myerson network allocation rule Y^m, using the extension method (2.13) formulated in the previous section.

Definition 3.1 *The* ***probabilistic Myerson value*** *is defined as the probabilistic network allocation rule* $\Psi^m\colon \mathbb{W}^N \times \mathbb{P}^N \to \mathbb{R}^N$ *such that for every player* $i \in N$*, every multilinear probabilistic network game* $v_w \in \mathbb{W}^N$ *with associated network game* $w \in \mathbb{H}^N$*, and every probabilistic network* $p \in \mathbb{P}^N$*:*

$$\Psi^m_i(v_w, p) = \sum_{g \subseteq g(p)} Y^m_i(g,w) \cdot \mu(g,p) = \sum_{g \subseteq g(p)} Y^m_i(g,w) \left(\prod_{ij \in g} p_{ij} \cdot \prod_{ij \notin g} (1 - p_{ij}) \right) \tag{3.1}$$

Clearly, for every $p \in \mathbb{P}^N$, the probabilistic Myerson value is the expectation of the Myerson network allocation rule Y^m for the induced probability distribution $\mu(\cdot, p)\colon \mathbb{G}^N \to [0,1]$. The following proposition shows that every probabilistic network game corresponds to a network game so that the probabilistic Myerson value of the probabilistic network game is equal to the Myerson value of the corresponding network game.

Proposition 3.2 *For each probabilistic network $p \in \mathbb{P}^N$ and the probabilistic network game $v_w \in \mathbb{W}^N$, there corresponds a network game $z : \mathbb{G}^N \to \mathbb{R}$ given by $z(g) = v_w(p^g)$ such that $\Psi_i^m(v_w, p) = Y_i^m(z, g(p))$.*

Proof. Given $p \in \mathbb{P}^N$, $g \in \mathbb{G}^N$, and $S \subseteq N \setminus i$, the probabilistic subnetworks $p^{g|_{S\cup i}}$ of $g|_{S\cup i}$ and $p^{g|_S}$ of $g|_S$ are obtained from the formula $p_{jk}^{g|_T} = \min\{p_{jk}, e_{jk}^{g|_T}\}$ for $T = \{S, S \cup i\}$ and $j, k \in N$. Then, from Lemma 2.10, we get

$$\begin{aligned} v_w(p^{g|_S}) &= \sum_{g' \subseteq g(p)} \mu(g', p) w(g' \cap g|_S) \\ v_w(p^{g|_{S\cup i}}) &= \sum_{g' \subseteq g(p)} \mu(g', p) w(g' \cap g|_{S\cup i}) \end{aligned}$$

Now, take $g = g(p)$ in particular. We have,

$$\begin{aligned} v_w(p^{g(p)|_S}) &= \sum_{g' \subseteq g(p)} \mu(g', p) w(g'|_S) \\ v_w(p^{g(p)|_{S\cup i}}) &= \sum_{g' \subseteq g(p)} \mu(g', p) w(g'|_{S\cup i}) \end{aligned}$$

Thus,

$$\begin{aligned} Y_i^m(z, g(p)) &= \sum_{S \subseteq N \setminus i} \left\{ z(g(p)|_{S\cup i}) - z(g(p)|_S) \right\} \frac{s!(n-s-1)!}{n!} \\ &= \sum_{S \subseteq N \setminus i} \left\{ v_w(p^{g(p)|_{S\cup i}}) - v_w(p^{g(p)|_S}) \right\} \frac{s!(n-s-1)!}{n!} \\ &= \sum_{S \subseteq N \setminus i} \left\{ \sum_{g' \subseteq g(p)} \mu(g', p) w(g'|_{S\cup i}) - \sum_{g' \subseteq g(p)} \mu(g', p) w(g'|_S) \right\} \frac{s!(n-s-1)!}{n!} \\ &= \sum_{g' \subseteq g(p)} \mu(g', p) \sum_{S \subseteq N \setminus i} \left\{ w(g'|_{S\cup i}) - w(g'|_S) \right\} \frac{s!(n-s-1)!}{n!} \\ &= \sum_{g' \subseteq g(p)} \mu(g', p) \cdot Y_i^m(g', w) \\ &= \Psi_i^m(v_w, p) \end{aligned} \tag{3.2}$$

This completes the proof of the assertion. ■

Prior to investigating the properties of the probabilistic Myerson value, we link this probabilistic network allocation rule to the well-known construction of the original Myerson value as a Shapley value of an appropriately constructed cooperative game as seminally set out by Myerson [13].

Cooperative games and the Myerson value. We recall that a *cooperative game* on the player set $N = \{1, \dots, n\}$ is a function $\sigma \colon 2^N \to \mathbb{R}$ that assigns a worth $\sigma(S)$ to every coalition $S \subseteq N$ such that $\sigma(\emptyset) = 0$. The function σ is also called a *characteristic function* of this game [15].

A vector $x \in \mathbb{R}^N$ is an *imputation* of the cooperative game σ if $\sum_{i \in N} x_i = \sigma(N)$. A well-known axiomatic rule that assigns to every game σ some well-constructed imputation $\phi(\sigma) \in \mathbb{R}^N$ was introduced by Shapley [16]. This *Shapley value* is defined by

$$\phi_i(\sigma) = \sum_{S \subseteq N \setminus \{i\}} \frac{(|S|)!(n - |S| - 1)!}{n!} \left[\sigma(S \cup \{i\}) - \sigma(S)\right]. \tag{3.3}$$

The Shapley value has been characterized through multiple sets of axioms. Shapley [16] introduced the first axiomatization of this value, which is founded on efficiency, symmetry, null player property and additivity/linearity.

With regard to the Myerson network allocation rule Y^m introduced before, we note that by comparing (3.3) and (2.2), clearly, for every network game $w \in \mathbb{H}^N$

$$Y_i^m(g, w) = \phi_i(\sigma_g^w) \tag{3.4}$$

where the restricted game is defined by

$$\sigma_g^w(S) = w(g|_S). \tag{3.5}$$

We can express the probabilistic Myerson value as the Shapley value of a certain well-constructed cooperative game, as the next proposition shows.

Proposition 3.3 *Given a multilinear probabilistic network game $v_w \in \mathbb{W}^N$ with associated network game $w \in \mathbb{H}^N$ and probabilistic network $p \in \mathbb{P}^N$, the probabilistic Myerson value $\Psi^m(v_w, p)$ is the Shapley value of the cooperative game $\sigma^p : 2^N \to \mathbb{R}$ given by*

$$\sigma^p(S) = \sum_{g \subseteq g(p)} w(g|_S) \cdot \mu(g, p) \tag{3.6}$$

Proof. Let $p \in \mathbb{P}^N$ be fixed. Using (3.5), we derive that

$$\sigma^p(S) = \sum_{g \subseteq g(p)} \sigma_g^w(S) \cdot \mu(g, p).$$

By the linearity property of the Shapley value and applying (3.4),

$$\phi_i(\sigma^p) = \sum_{g \subseteq g(p)} \phi_i\left(\sigma_g^w\right) \cdot \mu(g, p) = \sum_{g \subseteq g(p)} Y_i^m(g, w) \cdot \mu(g, p).$$

This completes the proof of the assertion. ■

3.1 An axiomatization of Ψ^m using equal bargaining power

Our first axiomatization of the probabilistic Myerson value extends the axiomatization of the Myerson network allocation rule of Jackson and Wolinsky [8] to our probabilistic framework. This calls for a definition of the axiom of equal bargaining power. Given $p \in \mathbb{P}^N$ and every $ij \in g_N$, we define the probabilistic network $p^{-ij} \in \mathbb{P}^N$ given by

$$p_{kl}^{-ij} = \begin{cases} p_{kl} & \text{if } kl \neq ij; \\ 0 & \text{if } kl = ij. \end{cases}$$

We are now able to define the axiom of equal bargaining power to our setting.

Definition 3.4 *A probabilistic network allocation rule $\Psi : \mathbb{W}^N \times \mathbb{P}^N \to \mathbb{R}^N$ satisfies **equal bargaining power** if for every $v_w \in \mathbb{W}^N$ with associated network game $w \in \mathbb{H}^N$ and every $p \in \mathbb{P}^N$:*

$$\Psi_i(v_w, p) - \Psi_i(v_w, p^{-ij}) = \Psi_j(v_w, p) - \Psi_j(v_w, p^{-ij}) \tag{3.7}$$

for every $i, j \in N$ with $i \neq j$.

In Jackson and Wolinsky [8], the two axioms of component balance and equal bargaining power are stated in the context of network games. Jackson and Wolinsky showed there that these two axioms are sufficient to characterize the Myerson network allocation rule Y^m.

Next, we first prove that the probabilistic Myerson value satisfies the two stated axioms. Then, we prove that the probabilistic Myerson value is the unique probabilistic network allocation rule satisfying these two axioms and, therefore, that these axioms provide a complete characterization of the probabilistic Myerson value. In doing so, we shall proceed following the methodology set out in Calvo et al. [5].

Proposition 3.5 *The probabilistic Myerson value satisfies the equal bargaining power property.*

Proof. Let $v_w \in \mathbb{W}^N$ be a probabilistic network game with associated network game $w \in \mathbb{H}^N$ and let $p \in \mathbb{P}^N$ be a probabilistic network. Let $i, j \in N$ such that $i \neq j$ and $p_{ij} > 0$. Define $\rho = \sigma^p - \sigma^{p^{-ij}} : 2^N \to \mathbb{R}$ as a derived game of coalitional differentials.

We outline the strategy of our proof. First, we show that for all $S \subseteq N \setminus \{i, j\}$, it holds that $\rho(S \cup \{i\}) = \rho(S \cup \{j\})$. This implies that i and j are symmetric in the cooperative game with characteristic function ρ. Therefore, by the symmetry property of the Shapley value [16], $\phi_i(\rho) = \phi_j(\rho)$ and, hence, $\phi_i(\sigma^p - \sigma^{p^{-ij}}) = \phi_j(\sigma^p - \sigma^{p^{-ij}})$. Now, applying the additivity axiom of the Shapley value stated in [16], $\phi_i(\sigma^p) - \phi_i(\sigma^{p^{-ij}}) = \phi_j(\sigma^p) - \phi_j(\sigma^{p^{-ij}})$. Finally, applying Proposition 3.3 completes the proof.

We first show that $\rho(S \cup \{i\}) = 0$. Analogously, it follows that $\rho(S \cup \{j\}) = 0$. Using the fact that if

$ij \in g$, then $p_{ij}^{-ij} = 0$ and hence $\mu\left(g, p^{-ij}\right) = 0$, we have the following:

$$\begin{aligned}
\rho(S \cup \{i\}) &= \sigma^{p}(S \cup \{i\}) - \sigma^{p^{-ij}}(S \cup \{i\}) \\
&= \sum_{g \subseteq g(p)} w(g|_{S \cup \{i\}}) \cdot \mu(g, p) - \sum_{g \subseteq g(p^{-ij})} w(g|_{S \cup \{i\}}) \cdot \mu\left(g, p^{-ij}\right) \\
&= \sum_{g \subseteq g(p)} w(g|_{S \cup \{i\}}) \cdot \mu(g, p) - \sum_{g \subseteq g(p)} w(g|_{S \cup \{i\}}) \cdot \mu\left(g, p^{-ij}\right) \\
&= \sum_{g \subseteq g(p)} w(g|_{S \cup \{i\}}) \left(\mu(g, p) - \mu\left(g, p^{-ij}\right) \right) \\
&= \sum_{g \subseteq g(p):\ ij \in g} w(g|_{S \cup \{i\}}) \left(\mu(g, p) - \mu\left(g, p^{-ij}\right) \right) + \\
&\qquad + \sum_{g \subseteq g(p):\ ij \notin g} w(g|_{S \cup \{i\}}) \left(\mu(g, p) - \mu\left(g, p^{-ij}\right) \right).
\end{aligned}$$

Recall that for all probabilistic networks $p \in \mathbb{P}^N$ we have that $\mu(g, p) = \prod_{ij \in g} p_{ij} \cdot \prod_{ij \notin g}(1 - p_{ij})$. Now, if $ij \notin g$,

$$\mu\left(g, p^{-ij}\right) = \frac{\mu(g, p)}{1 - p_{ij}}.$$

Hence, if $ij \notin g$ it holds that

$$\mu(g, p) - \mu\left(g, p^{-ij}\right) = \mu(g, p) - \frac{\mu(g, p)}{1 - p_{ij}} = \frac{-p_{ij}}{1 - p_{ij}} \mu(g, p). \tag{3.8}$$

Therefore,

$$\rho(S \cup \{i\}) = \sum_{g \subseteq g(p):\ ij \in g} w(g|_{S \cup \{i\}}) \mu(g, p) - \sum_{g \subseteq g(p):\ ij \notin g} w(g|_{S \cup \{i\}}) \frac{p_{ij}\, \mu(g, p)}{1 - p_{ij}}. \tag{3.9}$$

Now, the set of networks $\{g \subseteq g(p) \mid ij \in g\}$ can be written as $\{g + ij \mid g \subseteq g(p) \text{ and } ij \notin g\}$. Hence,

$$\rho(S \cup \{i\}) = \sum_{g \subseteq g(p):\ ij \notin g} w(g + ij|_{S \cup \{i\}}) \mu(g + ij, p) - \sum_{g \subseteq g(p):\ ij \notin g} w(g|_{S \cup \{i\}}) \frac{p_{ij}\, \mu(g, p)}{1 - p_{ij}}.$$

Clearly, $g + ij|_{S \cup \{i\}}$ is the same network as $g|_{S \cup \{i\}}$ given that the coalition $S \cup \{i\}$ does not include j. This in turn implies that

$$\begin{aligned}
\rho(S \cup \{i\}) &= \sum_{g \subseteq g(p):\ ij \notin g} w(g|_{S \cup \{i\}}) \mu(g + ij, p) - \sum_{g \subseteq g(p):\ ij \notin g} w(g|_{S \cup \{i\}}) \frac{p_{ij}\, \mu(g, p)}{1 - p_{ij}} \\
&= \sum_{g \subseteq g(p):\ ij \notin g} w(g|_{S \cup \{i\}}) \left[\mu(g + ij, p) - \frac{p_{ij} \cdot \mu(g, p)}{1 - p_{ij}} \right].
\end{aligned}$$

Furthermore, $\mu(g + ij, p) = \frac{\mu(g,p)\cdot p_{ij}}{1-p_{ij}}$. Therefore,

$$\rho(S \cup \{i\}) = \sum_{g \subseteq g(p):\ ij \notin g} w(g|_{S\cup\{i\}}) \left[\frac{p_{ij} \cdot \mu(g,p)}{1-p_{ij}} - \frac{p_{ij} \cdot \mu(g,p)}{1-p_{ij}} \right] = 0 \tag{3.10}$$

This completes the proof of the proposition. ■

Using the shown properties, we are able to state and prove the main, full characterization of the probabilistic Myerson value on the space of multilinear probabilistic network games $\mathbb{W}^N \subset \mathbb{V}^N$.

Theorem 3.6 *The probabilistic Myerson value Ψ^m is the unique probabilistic network allocation rule on $\mathbb{W}^N \times \mathbb{P}^N$ that satisfies component balance and equal bargaining power.*

Proof. Suppose that Ψ^1 and Ψ^2 are two probabilistic network allocation rules on $\mathbb{W}^N \times \mathbb{P}^N$ that satisfy component balance and equal bargaining power. As we have already proven that the probabilistic Myerson value satisfies component balance and equal bargaining power in Propositions 2.13 and 3.5, to prove uniqueness, we need to show that $\Psi^1 = \Psi^2$ on $\mathbb{W}^N \times \mathbb{P}^N$.
Let $v_w \in \mathbb{W}^N$ with associated network game $w \in \mathbb{H}^N$. First, consider the empty probabilistic network $p_0 \in \mathbb{P}^N$. Then obviously $\Psi^1(v_w, p_0) = \Psi^2(v_w, p_0) = 0$.
Next, suppose that $\Psi^1 \neq \Psi^2$ and let $p \in \mathbb{P}^N$ be such that p's support $g(p)$ has a minimum number of links such that $\Psi^1(v_w, p) \neq \Psi^2(v_w, p)$. By the minimality of $g(p)$, we know that for any link $ij \in g_N$ with $p_{ij} > 0$, it holds that $\Psi^1(v_w, p^{-ij}) = \Psi^2(v_w, p^{-ij})$.
Hence, by the equal bargaining power property of Ψ^1 and Ψ^2,

$$\begin{aligned}\Psi^1_i(v_w, p) - \Psi^1_j(v_w, p) &= \Psi^1_i(v_w, p^{-ij}) - \Psi^1_j(v_w, p^{-ij}) \\ &= \Psi^2_i(v_w, p^{-ij}) - \Psi^2_j(v_w, p^{-ij}) = \Psi^2_i(v_w, p) - \Psi^2_j(v_w, p).\end{aligned}$$

From this we derive that $\Psi^1_i(v_w, p) - \Psi^2_i(v_w, p) = \Psi^1_j(v_w, p) - \Psi^2_j(v_w, p)$ whenever i and j both belong to the same network component $N(h)$ where $h \in C(g(p))$. Thus, we can find numbers $D_h \in \mathbb{R}$ where $h \in C(g(p))$ such that $\Psi^1_i(v_w, p) - \Psi^2_i(v_w, p) = D_h$ for all $i \in N(h)$.
By component balance of both Ψ^1 and Ψ^2 on $\mathbb{W}^N \times \mathbb{P}^N$:

$$\sum_{i \in N(h)} \Psi^1_i(v_w, p) = \sum_{i \in N(h)} \Psi^2_i(v_w, p) = v_w(p^h)$$

Hence, we have

$$0 = \sum_{i \in N(h)} \Psi^1_i(v_w, p) - \sum_{i \in N(h)} \Psi^2_i(v_w, p) = \sum_{i \in N(h)} \left(\Psi^1_i(v_w, p) - \Psi^2_i(v_w, p) \right) = D_h \cdot |N(h)|$$

implying thereby that $D_h = 0$. Hence, $\Psi^1_i(v_w, p) = \Psi^2_i(v_w, p)$, showing the assertion. ■

3.2 An axiomatization of Ψ^m using balanced contributions

Given a probabilistic network $p \in \mathbb{P}^N$, define the probabilistic network $p^{-i} \in \mathbb{P}^N$ by

$$p_{kl}^{-i} = \begin{cases} p_{kl} & \text{if } kl \notin L_i(g(p)); \\ 0 & \text{if } kl \in L_i(g(p)). \end{cases}$$

The probabilistic network p^{-i} is the network derived from p in which all links with player i are completely severed. This allows us to introduce the following property.

Definition 3.7 *A probabilistic network allocation rule $\Psi \colon \mathbb{W}^N \times \mathbb{P}^N \to \mathbb{R}^N$ satisfies the **balanced contributions property** if for every $v_w \in \mathbb{W}^N$ with associated network game $w \in \mathbb{H}^N$ and every $p \in \mathbb{P}^N$:*

$$\Psi_i(v_w, p) - \Psi_i(v_w, p^{-j}) = \Psi_j(v_w, p) - \Psi_j(v_w, p^{-i}). \tag{3.11}$$

for all $i, j \in N$ with $i \neq j$.

We shall prove that the probabilistic Myerson value is the unique allocation rule on $\mathbb{W}^N \times \mathbb{P}^N$ that satisfies component balance as well as the balanced contributions property. To prove this, we use the balanced contributions property of the Myerson value of network games due to Slikker [18]. We begin by proving a property that is important in proving the subsequent proposition.

Lemma 3.8 *Let $p \in \mathbb{P}^N$. Then for every player $j \in N$ and for every subnetwork $g \subseteq g(p^{-j})$ it holds that*

$$\mu\left(g, p^{-j}\right) = \sum_{h_j \subseteq L_j(g(p))} \mu(g + h_j, p) \tag{3.12}$$

Proof. Given $p \in \mathbb{P}^N$, select any network $g \subseteq g\left(p^{-j}\right)$. Thus, g does not contain any link in $L_j(g(p))$. Hence, the set of links that do not belong to g can be divided into $L_j(g(p))$ and $g_N \setminus \left(g \cup L_j(g(p))\right)$. Hence,

$$\begin{aligned} \mu(g, p) &= \prod_{kl \in g} p_{kl} \times \prod_{kl \in g_N \setminus g} (1 - p_{kl}) \\ &= \prod_{kl \in g} p_{kl} \times \prod_{kl \in L_j(g(p))} (1 - p_{kl}) \times \prod_{kl \in g_N \setminus \left(g \cup L_j(g(p))\right)} (1 - p_{kl}) \end{aligned}$$

Similarly, we derive that

$$\mu\left(g, p^{-j}\right) = \prod_{kl \in g} p_{kl}^{-j} \times \prod_{kl \in L_j(g(p))} \left(1 - p_{kl}^{-j}\right) \times \prod_{kl \in g_N \setminus \left(g \cup L_j(g(p))\right)} \left(1 - p_{kl}^{-j}\right).$$

On the other hand,

$$p_{kl}^{-j} = \begin{cases} p_{kl} & \text{if } kl \notin L_j(g(p)); \\ 0 & \text{if } kl \in L_j(g(p)). \end{cases}$$

Hence,

$$\mu(g, p^{-j}) = \prod_{kl \in g} p_{kl} \times \prod_{kl \in g_N \setminus (g \cup L_j(g(p)))} (1 - p_{kl}) = \frac{\mu(g, p)}{\prod_{kl \in L_j(g(p))} (1 - p_{kl})}.$$

Next, for $h_j \subseteq L_j(g)$, $h_j \neq \varnothing$:

$$\mu(g + h_j, p) = \frac{\mu(g, p) \prod_{kl \in h_j} p_{kl}}{\prod_{kl \in h_j} (1 - p_{kl})}.$$

Therefore,

$$\mu(g, p) + \sum_{h_j \subseteq L_j(g(p)), h_j \neq \varnothing} \mu(g + h_j, p) = \mu(g, p) \left[1 + \sum_{h_j \subseteq L_j(g(p)), h_j \neq \varnothing} \frac{\prod_{kl \in h_j} p_{kl}}{\prod_{kl \in h_j} (1 - p_{kl})} \right]$$

$$= \mu(g, p) \left[1 + \sum_{h_j \subseteq L_j(g(p)), h_j \neq \varnothing} \frac{\prod_{kl \in h_j} p_{kl} \prod_{kl \in L_j(g(p)) \setminus h_j} (1 - p_{kl})}{\prod_{kl \in L_j(g(p))} (1 - p_{kl})} \right]$$

$$= \mu(g, p) \left[1 + \frac{\sum_{h_j \subseteq L_j(g(p)), h_j \neq \varnothing} \left(\prod_{kl \in h_j} p_{kl} \prod_{kl \in L_j(g(p)) \setminus h_j} (1 - p_{kl}) \right)}{\prod_{kl \in L_j(g(p))} (1 - p_{kl})} \right]. \tag{3.13}$$

Now, consider the probabilistic network $p^{-(g(p) - L_j(g(p)))} \in \mathbb{P}^N$ given by

$$p_{kl}^{-(g(p) - L_j(g(p)))} = \begin{cases} 0 & \text{if } kl \notin L_j(g(p)); \\ p_{kl} & \text{if } kl \in L_j(g(p)). \end{cases}$$

The only networks that materialize with positive probability under this probabilistic network are subsets of $L_j(g(p))$. Hence,

$$\sum_{h_j \subseteq L_j(g(p))} \mu\left(h_j, p^{-(g(p) - L_j(g(p)))}\right) = 1$$

implying that

$$\sum_{h_j \subseteq L_j(g(p)), h_j \neq \varnothing} \mu\left(h_j, p^{-(g(p) - L_j(g(p)))}\right) = 1 - \mu\left(\varnothing, p^{-(g(p) - L_j(g(p)))}\right).$$

This leads to the conclusion that

$$\sum_{h_j \subseteq L_j(g(p)), h_j \neq \varnothing} \left(\prod_{kl \in h_j} p_{kl} \prod_{kl \in L_j(g(p)) \setminus h_j} (1 - p_{kl}) \right) = 1 - \prod_{kl \in L_j(g(p))} (1 - p_{kl}) \tag{3.14}$$

From (3.13) and (3.14),

$$\sum_{h_j \subseteq L_j(g(p))} \mu(g+h_j,p) = \mu(g,p) + \sum_{h_j \subseteq L_j(g(p)), h_j \neq \emptyset} \mu(g+h_j,p)$$

$$= \mu(g,p)\left[1 + \frac{1-\prod_{kl \in L_j(g(p))}(1-p_{kl})}{\prod_{kl \in L_j(g(p))}(1-p_{kl})}\right]$$

$$= \frac{\mu(g,p)}{\prod_{kl \in L_j(g(p))}(1-p_{kl})} = \mu\left(g,p^{-j}\right)$$

This completes the proof of the assertion. ∎

Next, we prove that the probabilistic Myerson value satisfies the balanced contributions property.

Proposition 3.9 *The probabilistic Myerson value satisfies the balanced contributions property.*

Proof. Let $v_w \in \mathbb{W}^N$ be a probabilistic network game, where $w \in \mathbb{H}^N$ is the associated network game. Furthermore, let $p \in \mathbb{P}^N$ be any probabilistic network. Now, for any $i,j \in N$ with $i \neq j$,

$$\Psi_i^m(v_w,p) - \Psi_i^m(v_w,p^{-j}) = \sum_{g \subseteq g(p)} Y_i^m(g,w)\cdot\mu(g,p) - \sum_{g \subseteq g(p^{-j})} Y_i^m(g,w)\cdot\mu\left(g,p^{-j}\right)$$

$$= \sum_{g \subseteq g(p)-L_j(g(p))}\left[\sum_{h_j \subseteq L_j(g(p))} Y_i^m(g+h_j,w)\cdot\mu(g+h_j,p) - Y_i^m(g,w)\cdot\mu\left(g,p^{-j}\right)\right]$$

Using Lemma 3.8, we can write

$$\Psi_i^m(v_w,p) - \Psi_i^m(v_w,p^{-j}) =$$

$$= \sum_{g \subseteq g(p)-L_j(g(p))}\left[\sum_{h_j \subseteq L_j(g(p))} Y_i^m(g+h_j,w)\cdot\mu(g+h_j,p) - Y_i^m(g,w)\left(\sum_{h_j \subseteq L_j(g(p))}\mu(g+h_j,p)\right)\right]$$

$$= \sum_{g \subseteq g(p)-L_j(g(p))}\left[\sum_{h_j \subseteq L_j(g(p))} Y_i^m(g+h_j,w)\cdot\mu(g+h_j,p) - Y_i^m(g,w)\cdot\mu(g,p) - Y_i^m(g,w)\left(\sum_{h_j \subseteq L_j(g(p)):\ h_j \neq \emptyset}\mu(g+h_j,p)\right)\right]$$

$$= \sum_{g \subseteq g(p)-L_j(g(p))}\ \sum_{h_j \subseteq L_j(g(p)):\ h_j \neq \emptyset}\left[Y_i^m(g+h_j,w) - Y_i^m(g,w)\right]\cdot\mu(g+h_j,p).$$

Now, if $L_j(g(p)) \neq \emptyset$ this can be rewritten as

$$\Psi_i^m(v_w, p) - \Psi_i^m(v_w, p^{-j}) = \sum_{g \subseteq g(p): L_j(g(p)) \neq \emptyset} \left[Y_i^m(g, w) - Y_i^m(g - L_j(g), w) \right] \cdot \mu(g, p)$$

Moreover, if $L_j(g(p)) = \emptyset$, it follows that $g - L_j(g) = g$ and, hence, $Y_i^m(g, w) - Y_i^m(g - L_j(g), w) = 0$. Together with the above, this implies that

$$\Psi_i^m(v_w, p) - \Psi_i^m(v_w, p^{-j}) = \sum_{g \subseteq g(p)} \left[Y_i^m(g, w) - Y_i^m(g - L_j(g), w) \right] \cdot \mu(g, p).$$

Repeating this derivation for j, we also conclude that

$$\Psi_j^m(v_w, p) - \Psi_j^m(v_w, p^{-j}) = \sum_{g \subseteq g(p)} \left[Y_j^m(g, w) - Y_j^m(g - L_j(g), w) \right] \cdot \mu(g, p).$$

We know from the properties of the Myerson value of network games that for any $g \in \mathbb{G}^N$,

$$Y_i^m(g, w) - Y_i^m(g - L_j(g), w) = Y_j^m(g, w) - Y_j^m(g - L_i(g), w).$$

Hence, the assertion follows. ■

We are now in the position to provide a complete characterization of the probabilistic Myerson value based on the balanced contributions property:

Theorem 3.10 *The probabilistic Myerson value Ψ^m is the unique probabilistic network allocation rule on $\mathbb{W}^N \times \mathbb{P}^N$ that satisfies component balance and the balanced contributions property.*

Proof. We have already shown that the probabilistic Myerson value satisfies component balance and the balanced contributions property in Propositions 2.13 and 3.9. We now show that Ψ^m is actually the unique probabilistic network allocation rule that satisfies component balance and the balanced contributions property on $\mathbb{W}^N \times \mathbb{P}^N$. Let $p \in \mathbb{P}^N$ be fixed. We prove the theorem by induction on $\#g(p)$.

For $\#g(p) = 0$, (i.e., $p = p_0$), by Definition 2.7, any allocation rule assigns a payoff of zero to each player. Therefore, the properties are satisfied trivially.

Let $k \in \mathbb{N}$. Now, assume that for every $p' \in \mathbb{P}^N$ such that $\#g(p') = k$ the probabilistic Myerson value $\Psi^m(\cdot, p')$ is the unique probabilistic network allocation rule that satisfies component balance and the balanced contributions property on $\mathbb{W}^N$.

Next consider $p \in \mathbb{P}^N$ such that $\#g(p) = k+1$. Now, suppose $\Psi^1(\cdot, p)$ and $\Psi^2(\cdot, p)$ are two probabilistic network allocation rules that satisfy component balance and the balanced contributions property on $\mathbb{W}^N$. We now show that $\Psi^1(\cdot, p) = \Psi^2(\cdot, p)$.

Take $h \in C(g(p))$ and let $i, j \in N(h)$ be such that $i \neq j$.[5] Now, by the balanced contributions property applied to Ψ^1, we have for any $v_w \in \mathbb{W}^N$ with associated network game $w \in \mathbb{H}^N$:

$$\Psi_i^1(v_w, p) - \Psi_i^1(v_w, p^{-j}) = \Psi_j^1(v_w, p) - \Psi_j^1(v_w, p^{-i})$$

[5]If no two such players exist, then $p = p_0$ and hence $\Psi^1(\cdot, p) = \Psi^2(\cdot, p) = 0$.

which implies that

$$\Psi_i^1(v_w, p) - \Psi_j^1(v_w, p) = \Psi_i^1(v_w, p^{-j}) - \Psi_j^1(v_w, p^{-i}). \tag{3.15}$$

By the induction hypothesis, $\Psi_i^1(v_w, p^{-j}) = \Psi_i^2(v_w, p^{-j})$ as well as $\Psi_j^1(v_w, p^{-i}) = \Psi_j^2(v_w, p^{-i})$. Hence,

$$\Psi_i^1(v_w, p^{-j}) - \Psi_j^1(v_w, p^{-i}) = \Psi_i^2(v_w, p^{-j}) - \Psi_j^2(v_w, p^{-i}). \tag{3.16}$$

Finally, applying the balanced contributions property to Ψ^2:

$$\Psi_i^2(v_w, p) - \Psi_i^2(v_w, p^{-j}) = \Psi_j^2(v_w, p) - \Psi_j^2(v_w, p^{-i})$$

which, in turn, implies that

$$\Psi_i^2(v_w, p) - \Psi_j^2(v_w, p) = \Psi_i^2(v_w, p^{-j}) - \Psi_j^2(v_w, p^{-i}). \tag{3.17}$$

From (3.15)-(3.17), it follows that

$$\Psi_i^1(v_w, p) - \Psi_j^1(v_w, p) = \Psi_i^2(v_w, p) - \Psi_j^2(v_w, p)$$

and, therefore,

$$\Psi_i^1(v_w, p) - \Psi_i^2(v_w, p) = \Psi_j^1(v_w, p) - \Psi_j^2(v_w, p).$$

Thus, we can find numbers $D_h \in \mathbb{R}$ such that $\Psi_i^1(v_w, p) - \Psi_i^2(v_w, p) = D_h$ for all $i \in N(h)$. Clearly, this holds across all components $h \in C(g(p))$.
By component balance of both Ψ^1 and Ψ^2, it follows that

$$\sum_{i \in N(h)} \Psi_i^1(v_w, p) = \sum_{i \in N(h)} \Psi_i^2(v_w, p) = v_w(p^h).$$

Hence, we have

$$\begin{aligned} 0 &= \sum_{i \in N(h)} \Psi_i^1(v_w, p) - \sum_{i \in N(h)} \Psi_i^2(v_w, p) = \sum_{i \in N(h)} \left(\Psi_i^1(v_w, p) - \Psi_i^2(v_w, p) \right) \\ &= D_h \times |N(h)| \end{aligned}$$

implying thereby that $D_h = 0$. Hence, $\Psi_i^1(v_w, p) = \Psi_i^2(v_w, p)$. ■

4 The Probabilistic Position Value

In this section, we investigate the probabilistic network allocation rule that is based in the position network allocation rule Y^p introduced in Section 2. The next definition introduces this specific rule as an expected position value.

Definition 4.1 *The* ***probabilistic position value*** *is defined as the probabilistic network allocation rule* $\Psi^p\colon \mathbb{W}^N \times \mathbb{P}^N \to \mathbb{R}^N$ *such that for every player* $i \in N$*, every multilinear probabilistic network game* $v_w \in \mathbb{W}^N$ *with associated network game* $w \in \mathbb{H}^N$*, and every probabilistic network* $p \in \mathbb{P}^N$*:*

$$\Psi_i^p(v_w, p) = \sum_{g \subseteq g(p)} Y_i^p(g, w) \cdot \mu(g, p) = \sum_{g \subseteq g(p)} \left(Y_i^p(g, w) \times \prod_{ij \in g} p_{ij} \times \prod_{ij \notin g} (1 - p_{ij}) \right) \tag{4.1}$$

Similar to the probabilistic Myerson value, the probabilistic position value is defined as the expected payoff under the position network allocation rule over the induced probability distribution derived from the probabilistic network under consideration. Thus, in line with Proposition 3.2, we have the following equivalence result, which proof is a simple adaptation of the proof of Proposition 3.2.

Proposition 4.2 *For each probability network* $p \in \mathbb{P}^N$*, and the probabilistic network game* $v_w \in \mathbb{W}^N$*, there corresponds a network game* $z : \mathbb{G}^N \to \mathbb{R}$ *given by* $z(g) = v_w(p^g)$ *such that* $\Psi^p(v_w, p) = Y^p(z, g(p))$*.*

An axiomatization of the probabilistic position value is centred on the generalization of the balanced link contribution property:

Definition 4.3 *A probabilistic network allocation rule* $\Psi\colon \mathbb{W}^N \times \mathbb{P}^N \to \mathbb{R}^N$ *satisfies the* ***balanced link contribution property*** *if for all* $v_w \in \mathbb{W}^N$ *with associated network game* $w \in \mathbb{H}^N$*,* $p \in \mathbb{P}^N$ *and* $i, j \in N$ *with* $i \neq j$*:*

$$\sum_{jk \in L_j(g(p))} \left[\Psi_i(v_w, p) - \Psi_i(v_w, p^{-jk}) \right] = \sum_{ik \in L_i(g(p))} \left[\Psi_j(v_w, p) - \Psi_j(v_w, p^{-ik}) \right].$$

We first prove that the probabilistic position value on the class of all multilinear probabilistic network games $\mathbb{W}^N$ satisfies the property introduced above. Then, we extend Slikker's [18] characterization, proving that the probabilistic position value is the unique allocation rule on $\mathbb{W}^N \times \mathbb{P}^N$ satisfying component balancedness as well as the balanced link contribution property.

Proposition 4.4 *The probabilistic position value satisfies the balanced link contribution property.*

Proof. Let $v_w \in \mathbb{W}^N$ be a multilinear probabilistic network game with associated network game

$w \in \mathbb{H}^N$. Furthermore, let $p \in \mathbb{P}^N$ be fixed and $i, j \in N$ with $ij \in g(p)$. Then,

$$\sum_{jk \in L_j(g(p))} \left[\Psi_i^p(v_w, p) - \Psi_i^p\left(v_w, p^{-jk}\right) \right]$$

$$= \sum_{jk \in L_j(g(p))} \left[\sum_{g \subseteq g(p)} Y_i^p(g, w) \cdot \mu(g, p) - \sum_{g \subseteq g\left(p^{-jk}\right)} Y_i^p(g, w) \cdot \mu\left(g, p^{-jk}\right) \right]$$

$$= \sum_{jk \in L_j(g(p))} \left[\sum_{g \subseteq g(p)\colon jk \in g} Y_i^p(g, w) \cdot \mu(g, p) + \right.$$

$$\left. + \sum_{g \subseteq g(p)\colon jk \notin g} Y_i^p(g, w) \times \left(\mu(g, p) - \mu\left(g, p^{-jk}\right) \right) \right].$$

Now, first note that if $jk \in g$, it holds that $p_{jk}^{-jk} = 0$ and, hence, $\mu\left(g, p^{-jk}\right) = 0$. On the other hand, if $jk \notin g$, we have that $\mu\left(g, p^{-jk}\right) = \frac{\mu(g,p)}{1-p_{jk}}$. Hence,

$$\mu(g, p) - \mu\left(g, p^{-jk}\right) = \mu(g, p) - \frac{\mu(g, p)}{1 - p_{jk}} = \frac{-p_{jk}}{1 - p_{jk}}\, \mu(g, p).$$

Therefore,

$$\sum_{jk \in L_j(g(p))} \left[\Psi_i^p(v_w, p) - \Psi_i^p(v_w, p^{-jk}) \right]$$

$$= \sum_{jk \in L_j(g(p))} \left[\sum_{g \subseteq g(p)\colon jk \in g} Y_i^p(g, w) \cdot \mu(g, p) + \sum_{g \subseteq g(p)\colon jk \notin g} Y_i^p(g, w) \cdot \mu(g, p) \left(\frac{-p_{jk}}{1-p_{jk}} \right) \right].$$

Now, the set of networks $\{g \subseteq g(p) \mid jk \in g\}$ can be written as $\{g + jk \mid g \subseteq g(p) \text{ and } jk \notin g\}$. This allows us to conclude that

$$\sum_{jk \in L_j(g(p))} \left[\Psi_i^p(v_w, p) - \Psi_i^p\left(v_w, p^{-jk}\right) \right]$$

$$= \sum_{jk \in L_j(g(p))} \left[\sum_{g \subseteq g(p)\colon jk \notin g} Y_i^p(g + jk, w) \cdot \mu(g + jk, p) + \sum_{g \subseteq g(p)\colon jk \notin g} Y_i^p(g, w) \cdot \mu(g, p) \left(\frac{-p_{jk}}{1-p_{jk}} \right) \right].$$

On the other hand, $\mu(g+jk,p) = \frac{p_{jk}}{1-p_{jk}}\mu(g,p)$. Hence,

$$\sum_{jk\in L_j(g(p))} \left[\Psi_i^p(v_w,p) - \Psi_i^p\left(v_w,p^{-jk}\right) \right]$$

$$= \sum_{jk\in L_j(g(p))} \left[\sum_{g\subseteq g(p):\, jk\notin g} Y_i^p(g+jk,w)\cdot \frac{p_{jk}}{1-p_{jk}}\mu(g,p) + \sum_{g\subseteq g(p):\, jk\notin g} Y_i^p(g,w)\cdot \mu(g,p)\left(\frac{-p_{jk}}{1-p_{jk}}\right) \right]$$

$$= \sum_{jk\in L_j(g(p))} \left[\sum_{g\subseteq g(p):\, jk\notin g} \frac{p_{jk}}{1-p_{jk}}\mu(g,p)\left(Y_i^p(g+jk,w) - Y_i^p(g,w)\right) \right]$$

$$= \sum_{jk\in L_j(g(p))} \left[\sum_{g\subseteq g(p):\, jk\notin g} \mu(g+jk,p)\left(Y_i^p(g+jk,w) - Y_i^p(g,w)\right) \right]$$

$$= \sum_{jk\in L_j(g(p))} \left[\sum_{g\subseteq g(p):\, jk\in g} \mu(g,p)\left(Y_i^p(g,w) - Y_i^p(g-jk,w)\right) \right]$$

$$= \sum_{g\subseteq g(p)} \sum_{jk\in L_j(g)} \mu(g,p)\left(Y_i^p(g,w) - Y_i^p(g-jk,w)\right)$$

$$= \sum_{g\subseteq g(p)} \mu(g,p) \sum_{jk\in L_j(g)} \left(Y_i^p(g,w) - Y_i^p(g-jk,w)\right).$$

Similarly,

$$\sum_{ik\in L_i(g(p))} \left[\Psi_j^p(v_w,p) - \Psi_j^p\left(v_w,p^{-ik}\right) \right] = \sum_{g\subseteq g(p)} \mu(g,p) \sum_{ik\in L_i(g)} \left(Y_j^p(g,w) - Y_j^p(g-ik,w)\right).$$

By the balanced link contribution property of the position value for network games, we know that for all $g \subseteq g(p)$:

$$\sum_{jk\in L_j(g)} \left(Y_i^p(g,w) - Y_i^p(g-jk,w)\right) = \sum_{ik\in L_i(g)} \left(Y_j^p(g,w) - Y_j^p(g-ik,w)\right)$$

implying that

$$\sum_{jk\in L_j(g(p))} \left[\Psi_i^p(v_w,p) - \Psi_i^p\left(v_w,p^{-jk}\right) \right] = \sum_{ik\in L_i(g(p))} \left[\Psi_j^p(v_w,p) - \Psi_j^p\left(v_w,p^{-ik}\right) \right]$$

This shows that Ψ^p indeed satisfies the balanced link contribution property. ■

Finally, we are able to construct a characterization of the probabilistic position value founded on the two properties investigated above. The proof of the next theorem closely follows that of the original characterization in Slikker [17].

Theorem 4.5 *The probabilistic position value is the unique probabilistic network allocation rule on* $\mathbb{W}^N \times \mathbb{P}^N$ *that satisfies component balance and the balanced link contribution property.*

Proof. The probabilistic position value satisfies component balance and the balanced link contribution property on $\mathbb{W}^N \times \mathbb{P}^N$ from Propositions 2.13 and 4.4.
Conversely, suppose $\Psi \colon \mathbb{W}^N \times \mathbb{P}^N \to \mathbb{R}^N$ satisfies component balance and the balanced link contribution property. We intend to show that $\Psi = \Psi^p$. The proof is by induction on $\#g(p)$.

First, consider the empty probabilistic network $p_0 \in \mathbb{P}^N$, being the unique probabilistic network p with $\#g(p) = 0$. Then $\Psi(\cdot, p_0) = \Psi^p(\cdot, p_0) = 0$ follows from Definition 2.7 of Ψ.

Next, given $k \geq 1$, assume that $\Psi(\cdot, p') = \Psi^p(\cdot, p')$ for $p' \in \mathbb{P}^N$ with $\#g(p') = k - 1 \geqslant 0$. Take $v_w \in \mathbb{W}^N$ to be a multilinear probabilistic network game with associated network game $w \in \mathbb{H}^N$ and let $p \in \mathbb{P}^N$ be such that $\#g(p) = k \geq 1$.
Take any component $h \in C(g(p))$ and without loss of generality let $N(h) = \{1, 2, \ldots, m\}$. By the balanced link contributions property we have that for every $j \in N(h)$:

$$\sum_{jk \in L_j(g(p))} \left[\Psi_1(v_w, p) - \Psi_1\left(v_w, p^{-jk}\right) \right] = \sum_{1k \in L_1(g(p))} \left[\Psi_j(v_w, p) - \Psi_j\left(v_w, p^{-1k}\right) \right].$$

This implies that

$$\#L_j(g(p))\, \Psi_1(v_w, p) - \#L_1(g(p))\, \Psi_j(v_w, p) = \sum_{jk \in L_j(g(p))} \Psi_1\left(v_w, p^{-jk}\right) - \sum_{1k \in L_1(g(p))} \Psi_j\left(v_w, p^{-1k}\right).$$

But by the induction hypothesis,

$$\sum_{jk \in L_j(g(p))} \Psi_1\left(v_w, p^{-jk}\right) = \sum_{jk \in L_j(g(p))} \Psi_1^p\left(v_w, p^{-jk}\right)$$

as well as

$$\sum_{1k \in L_1(g(p))} \Psi_j\left(v_w, p^{-1k}\right) = \sum_{1k \in L_1(g(p))} \Psi_j^p\left(v_w, p^{-1k}\right)$$

Hence,

$$\#L_j(g(p))\, \Psi_1(v_w, p) - \#L_1(g(p))\, \Psi_j(v_w, p) = \sum_{jk \in L_j(g(p))} \Psi_1^p\left(v_w, p^{-jk}\right) - \sum_{1k \in L_1(g(p))} \Psi_j^p\left(v_w, p^{-1k}\right).$$

Furthermore, by component balance we have

$$\sum_{i=1}^{m} \Psi_i(v_w, p) = v_w(p^h).$$

Hence, we have a system of m equations in m unknowns. It is a straightforward exercise to show that this is a regular system in m variables $\Psi_1(v_w, p), \ldots, \Psi_m(v_w, p)$. Consequently, it has a unique solution. Since the position value satisfies balanced link contributions and component balance,

$\Psi_1^p(v_w, p), \ldots, \Psi_m^p(v_w, p)$ is a solution, and hence, it is the unique solution. We conclude that $\Psi(v_w, p) = \Psi^p(v_w, p)$ for $p \in \mathbb{P}^N$ with $\#g(p) = k$. ∎

5 Concluding remarks

In this paper we have extended the notion of network game of Jackson and Wolinsky [8] to that of a probabilistic network game, a framework where links are not formed deterministically but probabilistically. We focussed our analysis on the more restricted setting of multilinear probabilistic network games, a notion founded on the conception of Calvo et al. [5] where allocations are formulated as expectations, following a multilinear extension. We provide extensions as well as axiomatic characterizations of the two main fixed allocation rules, the Myerson value and the position value, using standard axioms.

We showed that axiomatizations of these two allocation rules founded on component balancedness can only be considered for the subclass of multilinear probabilistic network games $\mathbb{W}^N$. On this subclass $\mathbb{W}^N$, we have shown an equivalence between the probabilistic allocation rules and their deterministic network counterparts. However, we conjecture that such equivalence does not hold in general on the universal class of *all* probabilistic network games $\mathbb{V}^N$. It is yet an unresolved question how to define and characterize appropriate extensions of the probabilistic Myerson and position values on $\mathbb{V}^N$.

There are a number of alternative directions in which this research can be extended beyond $\mathbb{V}^N$. First, one can replace the restricted multilinear framework of Calvo et al. [5] by the general framework of Gomez et al. [7] where the independence assumption is dispensed with and any arbitrary probability distribution over the set of networks is considered. The analysis of the appropriate extensions of the Myerson and position values to this class of general probabilistic network games can be formulated around the more general framework founded on the extension of network games as set out by Gomez et al. in their analysis.

Second, probability measures are essentially additive and, therefore, their applicability in dealing with cooperative wealth generation is limited, especially in non-additive environments. *Fuzzy* measures being non-additive in general, can better model the interactions among players in more general terms. The interested reader may look at Li et al. [10]. Furthermore, multilinear extensions are used in designing fuzzy cooperative games as explored in Meng and Zhang [12] and Borkotokey et al. [2]. However, we have not found any instance of similar formulations in case of network-based approaches to wealth generation.

Third, we have restricted our analysis to the two main fixed allocation rules. Clearly, we can pursue the extension of the analysis to flexible network allocation rules—such as the link based and player based allocation rules in Jackson [9] in this framework.

References

[1] Billand, P., C. Bravard, S. S. Iyengar, R. Kumar and S. Sarangi. "Network connectivity under node failure." *Economics Letters* ,149 (2016): 164-167.

[2] Borkotokey, S., P. Hazarika and R. Mesiar (2015), "Fuzzy bi-cooperative games in multilinear extension form", *Fuzzy Sets and Systems*, 259, 44–55.

[3] Borkotokey, S., S. Sarangi and R. Kumar (2015), "A solution concept for network games: The role of multilateral interactions", *European Journal of Operational Research*, 243(3), 912–920.

[4] Borm, P., G. Owen and S. Tijs (1992), "On the position value for communication situations", *SIAM Journal on Discrete Mathematics*, 5, 305–320.

[5] Calvo, E., J. Lasaga and A. van den Nouweland (1999), "Values of games with probabilistic graphs", *Mathematical Social Sciences*, 37, 79–95.

[6] Ghintran, A., E. González–Arangüena and C. Manuel (2012), "A probabilistic position value", *Annals of Operations Research,* 201, 183–196.

[7] Gomez, D., E. González–Arangüena and, C. Manuel and G. Owen (2008), "A value for generalized probabilistic communication situations, *European Journal of Operations Research*, 190, 539–556.

[8] Jackson, M. and A. Wolinsky (1996) "A strategic model of social and economic networks", *Journal of Economic Theory*, 71 (1), 44–74.

[9] Jackson, M. (2005), "Allocation rules for network games", *Games and Economic Behaviour*, 51, 128–154.

[10] Li, C., M. Wang, and H. Wang (2011), "The fuzzy bargaining sets for simple fuzzy network games", *Eighth International Conference on Fuzzy Systems and Knowledge Discovery (FSKD)*, IEEE Proceedings, 1.

[11] Meessen, R. (1988), "Communication games", Masters' Thesis, Department of Mathematics, University of Nijmegen, The Netherlands.

[12] Meng, F.Y., and Q. Zhang (2010), "The Shapley function for fuzzy cooperative games with multilinear extension form", *Applied Mathematics Letters*, 23 (5), 644–650.

[13] Myerson, R. (1977), "Graphs and cooperation in games", *Mathematics of Operations Research*, 2, 225–229.

[14] Owen, G. (1968), *Game theory*, 4th edition, 2013, Emerald Publishing Group, Bingley, UK.

[15] Owen, G. (1972), "Multilateral extensions of games", *Management Science*, 18, 64–79.

[16] Shapley, L.S. (1953), "A value for n-person games". In H. Kuhn, & A.W. Tucker, (Eds), *Contributions to the Theory of Games II*, 307–317, Princeton University Press, Princeton, NJ.

[17] Slikker, M. (2005), "A characterization of the position value", *International Journal of Game Theory*, 33, 505–514.

[18] Slikker, M. (2007), "Bidding for surplus in network allocation problems", *Journal of Economic Theory*, 33, 505–514.

[19] van den Nouweland, A. and M. Slikker (2012), "An axiomatic characterization of the position value for network situations", *Mathematical Social Sciences*, 64, 266–271.