Gately Values of Cooperative Games*

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In memory of Stef Tijs (1937 – 2023)

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Abstract

We investigate Gately's solution concept for cooperative games with transferable utilities. Gately's conception introduced a bargaining solution that minimises the maximal quantified "propensity to disrupt" the negotiation process of the players over the allocation of the generated collective payoffs. We show that Gately's solution concept is well-defined for a broad class of games and that it can be interpreted as a compromise solution. We also consider a generalisation based on a parameter-based quantification of the propensity to disrupt. We provide an axiomatic characterisation of the original Gately value as well as these generalised Gately values. Furthermore, we investigate the relationship of these Gately values with the Core and the Nucleolus and show that Gately's solution is in the Core for all regular 3-player games, but is fundamentally different from the Nucleolus. We identify exact conditions under which these Gately values are Core imputations for arbitrary regular cooperative games. Finally, we investigate the relationship of the Gately value with the Shapley value.

Keywords: Game Theory; cooperative game; sharing value; Gately point; Core.

JEL classification: C71

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1 Introduction: Gately's solution concept

Gately (1974) seminally considered an allocation method founded on individual players' opportunities to disrupt the negotiations regarding the allocation of the generated collective payoffs. This conception is akin to the underlying logic of the Core (Gillies, 1959) of a cooperative game: The Core is founded on the idea that coalitions of players would threaten to abandon the negotiations over the allocation of the total generated collective worth, if the offered allocation does not at least assign each coalition's worth in the cooperative game.[1] Gately's conception refines this and formalises *to what extend* coalitions would disrupt such negotiations.

Gately (1974) introduced the notion of an individual player's *propensity to disrupt*, expressing the relative disruption an individual player causes when leaving the negotiations. In fact, Gately formulated this "propensity to disrupt" as the ratio of the other players' collective loss and the individual player's loss due to disruption of the negotiations. Gately's solution method aims to minimise the maximal propensity to disrupt over all imputations and players in the game. Staudacher and Anwander (2019) show that for most cooperative games this solution method results in a unique imputation, being the *Gately value* of the game under consideration.

Clearly, Gately's solution concept falls within the category of a *bargaining-based* solution concepts (Maschler, 1992) that also encompasses, e.g., the bargaining set (Aumann and Maschler, 1964), the Kernel (Davis and Maschler, 1965), and the Nucleolus (Schmeidler, 1969).[2] Contrary to many of these bargaining-based solution concepts, Gately's conception results in an easily to compute allocation rule that can also be categorised as a *compromise value* such as the CIS-value (Driessen and Funaki, 1991) and the τ-value (Tijs, 1981). These solution concepts have a fundamentally different foundation than the *axiomatic* allocation rules such as the egalitarian solution, the Shapley value (Shapley, 1953), the Banzhaf value (Banzhaf, 1965; Lehrer, 1988), and related notions.

Gately (1974) investigated his conception in the setting of one particular 3-player cost game only. Gately's notion was extended to arbitrary n-player cooperative games by Littlechild and Vaidya (1976). [3] Charnes et al. (1978) introduced various concepts that are closely related to an extended notion of the propensity to disrupt. They introduced *mollifiers* and *homomollifiers*, measuring the disparities emerging from abandonment of negotiations as differences rather than ratios. These formulations result in associated games with a given cooperative game. Charnes et al. (1978) primarily investigated the properties of these associated games.

Gately points: Existence, uniqueness and relationship with the Core Staudacher and Anwander (2019) point out that the original research questions as posed by Gately (1974) were never properly investigated and answered in the literature. In particular, Staudacher and Anwander focussed on one particular application within the broad range of possibilities in Gately's approach, namely the so-called *Gately point*—defined as an imputation in which all propensities to disrupt

[1]It is, therefore, implicitly assumed that a coalition can generate its worth independently of the actions taken by players and coalitions outside that coalition.

[2]For an overview of these solution concepts we also refer to textbooks such as Moulin (1986, 2004), Maschler et al. (2013), Owen (2013) and Gilles (2010).

[3]Littlechild and Vaidya used their extended notion of propensity to disrupt to define the *Disruption Nucleolus* in which these propensities are lexicographically minimised.

are balanced and minimal. The Gately point is a solution to a minimax problem and Staudacher and Anwander show that every standard cooperative game has a unique Gately point. This settles indeed the most basic question concerning Gately's original conception.

Here, we provide a natural and constructive axiomatic characterisation of the Gately value based on three simple properties. These properties relate to the properties introduced by Tijs (1987) to characterise the τ-value for quasi-balanced cooperative games. We modify these properties for the class of regular cooperative games to fully axiomatise the Gately value for this class of games. This shows in detail that the Gately value is actually a compromise value on the class of regular cooperative games.

Furthermore, we introduce the *dual Gately value* as the Gately point of the dual of a given cooperative game. We show that the dual Gately value is identical to the Gately value for the broad class of regular games. Hence, the Gately value is self-dual.

Gately's definition of his propensity to disrupt puts equal weight on assessing the loss or gain of the other players versus the loss or gain of the player under consideration. We consider a parametric formulation in which a weight is attached to the relative importance of the gain or loss of the individual player in comparison with the weight attached to the gain or loss of all other players in the game. The higher the assigned weight, the more an individual's loss or gain due to disruption is taken into account.

The imputations that balance these weighted propensities to disrupt are now referred to as generalised α-Gately points, where $\alpha > 0$ is the weight put on an individual's loss or gain due to disruption. It is clear that $\alpha = 1$ refers to the original Gately point. We show that for all $\alpha > 0$, all regular cooperative games admit a unique α-Gately point, generalising the insight of Staudacher and Anwander (2019).

Analysis of the Gately value in relation to other solution concepts Gately (1974) states clearly that he views the conception of Gately points and related concepts based on his notion of "propensity to disrupt" as leading to Core selectors. This is exemplified by the underlying conception of Gately's solution method as a Core-based bargaining process. In particular, this is supported by the analysis of the cost games considered by Gately (1974), which have rather large Cores. Here, we primarily explore the interesting and yet unexplored relationship between Gately points and the Core. In particular, we show that the unique Gately point is a Core selector for *every* regular 3-player cooperative game. Littlechild and Vaidya (1976) already showed that this cannot be extended to n-player games by constructing a 4-player game in which the Gately point is not in the Core.

We extend our analysis to α-Gately points and show that, for any $\alpha > 0$, the unique α-Gately point is in the Core of the game if and only if the game satisfies α-Top Dominance, a parametric variant of the top convexity condition. Particularly, the Top Dominance condition reduces to *top convexity* for zero-normalised games (Shubik, 1982; Jackson and van den Nouweland, 2005). Also, we show that the α-Top Dominance condition implies that the game is regular as well as partitionally superadditive. However, counterexamples show that there exist superadditive games with non-empty Cores that do not contain any α-Gately point.

The axiomatic solution concept seminally introduced by Shapley (1953) is now widely accepted as the prime value for cooperative games. It has resulted in a vast literature on determining the Shapley value and its properties on certain classes of cooperative games such as communication situations (Myerson, 1977, 1980), network games (Jackson and Wolinsky, 1996), and hierarchical permission structures (Gilles et al., 1992; Gilles and Owen, 1999). This indicates the validity of the question for which subclasses of cooperative games an alternative solution concept is equivalent to the Shapley value.

We investigate the equivalence of the Shapley and Gately values. We conclude that, indeed, for certain classes of regular games the Gately value results in exactly the same imputation as the Shapley value. This includes the class of cooperative games generated by unanimity games of equal-sized coalitions, the so-called k-games (van den Brink et al., 2023). Other classes of highly regular games also possess this equivalence property, showing that potentially for many subclasses of highly regular cooperative games these values might coincide.

In particular, the Gately and Shapley values are equivalent on the specific subclass of 2-games. This equivalence enables us to utilise the characterisation developed in van den Brink et al. (2023, Theorem 1) to establish an axiomatisation of the Gately value within this particular relevant subclass of regular games. In fact, the class of 2-games encompasses the binary cooperation on networks, where binary value-generating activity occurs on the links forming the network. The insight that many values are equivalent on this particular subclass of 2-games is relevant in the analysis of these games.

Structure of the paper We introduce and illustrate Gately's approach through an application to an exchange or "household" economy with three traders in Section 2. Section 3 develops the formal treatment of Gately's approach, defines generalised Gately points and values, and discusses the dual of the Gately value. This section concludes with a natural and simple axiomatisation of the Gately value. Section 4 is devoted to the investigation of the relationship of Gately points, the Core, the Nucleolus and other Core selectors for 3-player games. We conclude the paper in Section 5 with the investigation of the Gately value with the Core for arbitrary n-player games and the equivalence of the Gately and Shapley values. The final section summarises and considers further developments and research directions.

2 An illustrative example: A household economy

To illustrate the ideas behind Gately (1974)'s conception, we consider an exchange economy with three traders. We use the examples of value allocations in general exchange economies developed in Shafer (1980), Yannelis (1983) and Scafuri and Yannelis (1984). We particularly focus on the example developed in Shafer (1980) and Scafuri and Yannelis (1984). This example illustrates the intricacies of applying cooperative value concepts to exchange economies founded on the value theory set out by Shapley (1969) and Aumann (1975). The objective of these contributions is to apply to notion of the Shapley Value (Shapley, 1953) to identify "value allocations" in the economy. This is pursued by constructing a cooperative game with side payments from the economy, founded on the hypothesis that all generated utilities from commodity consumption are *cardinal* and transfers of these utilities can be considered as well as applied. As such, the exchange economy becomes a collective entity to which individual agents as members can make contributions. Allocated payoffs from the collectively generated wealth in the economy can now be supported through appropriate allocation of commodity bundles.[4]

The example developed by Shafer (1980) shows that, even if a trader is endowed with a zero commodity bundle, i.e., has no endowment, that trader can be assigned a strictly positive commodity bundle under the application of the Shapley value. This is due to the trader's role in the generation of the resulting gains from trade, due to the properties of her cardinal utility function. It shows that the Shapley value can lead to unnatural commodity allocations in such economies.

In the next discussion, we explicitly interpret the underlying exchange economy as a household of which its members make contributions to the collective well-being in the household. Endowment of goods act as inputs to the generation of such collective well-being. This well-being is contributed by every household member through application of a cardinal contribution function. The total well-being is subsequently allocated to the members through the appropriate allocation of quantities of commodities contributed to the household.

A household economy We consider a pure exchange economy with three agents—denoted as 0, 1 and 2—and two tradable commodities, X and Y. Each agent i is endowed with a cardinal utility function $u_i \colon \mathbb{R}^2_+ \to \mathbb{R}_+$ and a commodity bundle $e_i \in \mathbb{R}^2_+$.

We can interpret the three agents forming a household in which the three cardinal utility functions represent individual contribution functions for the three household members toward the household's collective well-being. A commodity bundle $(x, y) \in \mathbb{R}^2_+$ is converted by household member i in a contribution of $u_i(x, y) \geqslant 0$ to the collective well-being in the household.

In particular, it is assumed that all three household members have equal weight and that the total generated well-being in the household is, therefore, represented by the total household well-being function $U = u_0 + u_1 + u_2 \colon \left(\mathbb{R}^2_+\right)^3 \to \mathbb{R}_+$. It is assumed that the total well-being can be allocated among the three household members through the assignment of appropriate commodity bundles.

Now, an *allocation* in this household economy is a triple of bundles $x = (x_0, x_1, x_2) \in \left(\mathbb{R}^2_+\right)^3$ such that $x_0 + x_1 + x_2 = e = e_0 + e_1 + e_2$. The bundle x_i is assigned to household member i and contributes to generate the total well-being given by $u_0(x_0) + u_1(x_1) + u_2(x_2)$.

Using the simplified example studied by Scafuri and Yannelis (1984), we let the three cardinal utility functions[5] and endowments be given by

$$u_0(x_0, y_0) = \left[\tfrac{1}{2}x_0^{\alpha} + \tfrac{1}{2}y_0^{\alpha}\right]^{\frac{1}{\alpha}} \quad \text{and } e_0 = (0,0) \tag{1}$$

$$u_1(x_1, y_1) = \left[\tfrac{1}{2}x_1^{\beta} + \tfrac{1}{2}y_1^{\beta}\right]^{\frac{1}{\beta}} \quad \text{and } e_1 = (1,0) \tag{2}$$

$$u_2(x_2, y_2) = \left[\tfrac{1}{2}x_2^{\beta} + \tfrac{1}{2}y_2^{\beta}\right]^{\frac{1}{\beta}} \quad \text{and } e_2 = (0,1) \tag{3}$$

[4] We emphasise that in this construction, utility functions are used as generators of contributions to collectively generated wealth *as well as* traditional utility functions measuring individuals' well-being from consuming allocated commodity bundles.

[5] These utility functions are from the well-known family of "Constant Elasticity of Substitution" (CES) functions.

where $0 < \beta < \alpha < 1$ are the parameters that determine the efficiency of the contribution of a member to the collective household. The total endowment of the household is $e = e_0 + e_1 + e_2 = (1, 1)$. This configuration implies that agent 0 is more productive in use value generation than the other two members, but is not endowed with any of the two commodities.

Converting the household economy into a cooperative game Using the methodology set out in Shafer (1980) and Scafuri and Yannelis (1984), we can convert this household economy to a cooperative game with side payments $v \colon 2^N \longrightarrow \mathbb{R}_+$ based on the cardinal utility functions of the three members as use value generating functions under equal weighting of the members with for every coalition $S \subseteq N = \{0, 1, 2\}$:

$$v(S) = \max_{(x_0, x_1, x_2) \in \left(\mathbb{R}^2_+\right)^3} \left\{ \sum_{i \in S} u_i(x_i) \,\middle|\, \sum_{i \in S} (x_i - e_i) = 0 \right\} \tag{4}$$

An allocation $x \in \left(\mathbb{R}^2_+\right)^3$ *supports* a utility distribution $u = (u_0, u_1, u_2) \in \mathbb{R}^3$ with $u_0 + u_1 + u_2 = v(N)$ if $u_i(x_i) = u_i$ for all $i = 0, 1, 2$. Furthermore, an allocation $x \in \left(\mathbb{R}^2_+\right)^3$ is a *Core allocation* if for every coalition $S \in 2^N \colon \sum_{i \in S} u_i(x_i) \geqslant v(S)$.

From the utility functions and endowments given above, it is easy to establish that the corresponding cooperative game v is given by $v(0) = 0$, $v(1) = v(2) = 2^{-\frac{1}{\beta}}$, $v(01) = v(02) = 2^{-\frac{1}{\alpha}}$, $v(12) = 1$, and $v(N) = v(012) = 1$.[6]

Scafuri and Yannelis (1984) established that the Shapley value of this cooperative game is given by

$$\varphi(v) = \left(s_0 \, , \, \tfrac{1-s_0}{2} \, , \, \tfrac{1-s_0}{2} \right) \qquad \text{with } s_0 = \tfrac{1}{3} \left(2^{-\frac{1}{\alpha}} - 2^{-\frac{1}{\beta}} \right) \tag{5}$$

which is supported by the Shapley allocation $x^s_0 = (s_0, s_0)$ and $x^s_1 = x^s_2 = \left(\tfrac{1-s_0}{2} \, , \, \tfrac{1-s_0}{2} \right)$.

Remark that this Shapley allocation is not a Core allocation, since the coalition $12 = \{1, 2\}$ is only allocated $u_1(x^s_1) + u_2(x^s_2) = \varphi_1(v) + \varphi_2(v) = 1 - s_0 < 1 = v(12)$. In particular, we note that the Core of the cooperative game v is determined as

$$C(v) = \left\{ (0, t, 1-t) \,\middle|\, 2^{-\frac{1}{\alpha}} \leqslant t \leqslant 1 - 2^{-\frac{1}{\beta}} \right\}$$

The quintessential feature of this particular household economy is that Agent 0 does not contribute any commodity inputs to the household. Nevertheless, this dummy member has a superior value-generating technology to contribute to the collective household in comparison with the other two members of the household. This, in turn, is recognised in Shapley's conception, resulting in a strictly positive Shapley value for Agent 0 and, thus, the assignment of a strictly positive commodity bundle for that member in the identified Shapley allocation. This seems contradictory as is shown by the fact that this Shapley allocation is not a Core allocation in this household. We show that Gately's conception of his value corrects this problem and actually assigns a Core allocation to this household.

[6] Here, for convenience, we use the abbreviated notation $0 = \{0\} \subseteq N$, $01 = \{0, 1\} \subseteq N$, etc.

Gately's conception Gately (1974) introduced the idea that during any negotiation between the three household members 0,1 and 2, each of these three parties can disrupt the proceedings by departing the negotiations and that this disruption can be quantified. Gately explicitly introduced his conception to delineate and focus on a specific Core selector in the 3-player application explored in Gately (1974).[7] Gately's methodology is different from the foundations of other well-known bargaining solutions such as the Bargaining Set, the Kernel (Davis and Maschler, 1965) and the Nucleolus (Schmeidler, 1969). In particular, Gately devised a much simpler solution than the Nucleolus, which is notoriously hard to compute (Maschler, 1992).

Gately founded his approach on the mathematical quantification of the disruption of the negotiation process that each individual player can cause, represented by a player's "propensity to disrupt". Gately (1974, p. 200–201) introduces this concept as "the ratio of how much the two other players would lose if a player would refuse to cooperate to how much that player would lose if it refused to cooperate".

For Gately's conception we consider any imputation of the cooperative game v. Here, an *imputation* refers to any wealth distribution $u = (u_0, u_1, u_2) \in \mathbb{R}^3$ that is individually rational (IR), i.e., $(u_0, u_1, u_2) \geqslant \left(0, 2^{-\frac{1}{\beta}}, 2^{-\frac{1}{\beta}}\right)$, as well as efficient, i.e., $u_0 + u_1 + u_2 = v(012) = 1$.

If the bargaining household members consider a proposed IR and efficient imputation (u_0, u_1, u_2), Gately's notion of the propensity to disrupt by agent 0 would then be the ratio of the other agents' potential loss $u_1 + u_2 - v(12)$ to Agent 0's potential loss from non-cooperation, computed as $u_0 - v(0)$. Hence, using $u_0 + u_1 + u_2 = v(N) = 1$, Agent 0's propensity to disrupt is computed as

$$d_0(u_0, u_1, u_2) = \frac{u_1 + u_2 - v(12)}{u_0 - v(0)} = \frac{u_1 + u_2 - 1}{u_0} = \frac{-u_0}{u_0} = -1.$$

Similarly, we construct the propensity to disrupt for both other household members as

$$d_1(u_0, u_1, u_2) = \frac{u_0 + u_2 - v(02)}{u_1 - v(1)} = \frac{u_0 + u_2 - 2^{-\frac{1}{\alpha}}}{u_1 - 2^{-\frac{1}{\beta}}} = \frac{1 - 2^{-\frac{1}{\alpha}} - u_1}{u_1 - 2^{-\frac{1}{\beta}}} = \frac{1 - 2^{-\frac{1}{\alpha}} - 2^{-\frac{1}{\beta}}}{u_1 - 2^{-\frac{1}{\beta}}} - 1$$

$$d_2(u_0, u_1, u_2) = \frac{u_0 + u_1 - v(02)}{u_2 - v(2)} = \frac{u_0 + u_1 - 2^{-\frac{1}{\alpha}}}{u_2 - 2^{-\frac{1}{\beta}}} = \frac{1 - 2^{-\frac{1}{\alpha}} - u_2}{u_2 - 2^{-\frac{1}{\beta}}} = \frac{1 - 2^{-\frac{1}{\alpha}} - 2^{-\frac{1}{\beta}}}{u_2 - 2^{-\frac{1}{\beta}}} - 1$$

Gately's motivation was that, if a player would get a relatively small payout, the player's propensity to disrupt the agreement is relatively high, indicating a higher impact on the negotiation.[8] Now, the stated objective of Gately's proposed solution is to select an imputation that minimises the maximal propensity to disrupt at that imputation. Hence, one should select an imputation that solves the

[7]From the stated methodology, Gately's focus is on how individual players negotiate on the allocation of collectively generated worth. The methodology is similar to the definition of the Core as those allocations to which there are no objections, in the sense that there are no coalitional incentives to abandon the negotiations to allocate the collectively generated worth by pursuing an alternative arrangement.

[8]In particular, if a player would not get any benefit in the negations in the sense that $u_i = v(i)$, her propensity to disrupt is infinitely large. Similarly, if the player would be proposed to receive the total generated benefit $u_i = v(N)$, her propensity to disrupt is usually negative.

minimax problem stated as

$$\min_{(u_0,u_1,u_2)\geqslant\left(0,2^{-\frac{1}{\beta}},2^{-\frac{1}{\beta}}\right):\ u_0+u_1+u_2=1} \max\ \{\ d_0(u_0,u_1,u_2), d_1(u_0,u_1,u_2), d_2(u_0,u_1,u_2)\ \}\ .$$

Noting that $d_1(u_0,u_1,u_2) > -1$ and $d_2(u_0,u_1,u_2) > -1$, this leads to the condition that $d_1(u_0,u_1,u_2) = d_2(u_0,u_1,u_2)$, since $d_0(u_0,u_1,u_2) = -1$ is certainly not a maximum. Therefore, $u_0 = 0$. Hence, Gately's solution is determined by a system of two equations: $u_1 = u_2$ and $u_1 + u_2 = 1$. Therefore, the solution to Gately's minimax problem is unique and determined as $g(v) = (0, \frac{1}{2}, \frac{1}{2})$, supported by the Gately allocation $x^g = (x_0^g, x_1^g, x_2^g) = \left(\ (0,0)\ , \left(\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right)\ \right)$. We remark that x^g is indeed a Core allocation for this particular household economy.

We remark that Gately's solution in this household economy satisfies the equal treatment property and is coalitionally fair. We refer to Section 3.1 for a further discussion of the general fairness of Gately's solution for cooperative games.

Generalising Gately's solution conception Next, we assume that a player can discount her own losses due to disruption or, conversely, assign more weight to her own losses than the losses of the other players in a reformulation of Gately's propensity to disrupt. In particular, we introduce a weight parameter $\gamma > 0$ for the denominator in the propensity to disrupt. Instead of applying this directly to the formulated propensity to disrupt d_i, $i = 0, 1, 2$, itself, we apply this weight in the modified form $d_i + 1$, denoted as d_i^γ. Hence, for each of the three agents in the household economy, we introduce the γ-weighted propensity to disrupt as

$$d_0^\gamma(u_0,u_1,u_2) = 0 \qquad d_1^\gamma(u_0,u_1,u_2) = \frac{1-2^{-\frac{1}{\alpha}}-2^{-\frac{1}{\beta}}}{\left(u_1-2^{-\frac{1}{\beta}}\right)^\gamma} > 0 \qquad d_2^\gamma(u_0,u_1,u_2) = \frac{1-2^{-\frac{1}{\alpha}}-2^{-\frac{1}{\beta}}}{\left(u_2-2^{-\frac{1}{\beta}}\right)^\gamma} > 0$$

For any $\gamma > 0$, a generalised Gately solution would solve the minimax problem given by

$$\min_{(u_0,u_1,u_2)\geqslant\left(0,2^{-\frac{1}{\beta}},2^{-\frac{1}{\beta}}\right):\ u_0+u_1+u_2=1} \max\left\{\ d_0^\gamma(u_0,u_1,u_2), d_1^\gamma(u_0,u_1,u_2), d_2^\gamma(u_0,u_1,u_2)\ \right\}\ .$$

The generalised solution for this modified Gately conception is now determined by the equations:

$$\frac{1-2^{-\frac{1}{\alpha}}-2^{-\frac{1}{\beta}}}{\left(u_1-2^{-\frac{1}{\beta}}\right)^\gamma} = \frac{1-2^{-\frac{1}{\alpha}}-2^{-\frac{1}{\beta}}}{\left(u_2-2^{-\frac{1}{\beta}}\right)^\gamma} \qquad \text{and} \qquad u_1+u_2=1 \text{ with } u_0=0.$$

The equations stated above lead to the conclusion that, for every $\gamma > 0$, the generalised Gately solution is identical to the regular Gately value, i.e., $g^\gamma(v) = g(v) = (0, \frac{1}{2}, \frac{1}{2})$ for all $\gamma > 0$.

Again, there is a close relationship between these (generalised) Gately solutions and the Core of this household economy in the sense that every generalised Gately solution is in the Core as well.

This close relationship between these Gately values and the Core refers directly to Gately (1974)'s original motivation to identify his solutions as Core selectors.

Finally, we remark that in this application the generalised γ-Gately value for any $\gamma > 0$ including the regular Gately value coincides with both the τ-value (Tijs, 1981) and the Nucleolus (Schmeidler, 1969). Moreover, in this application the Gately value is obviously coalitionally fair and satisfies the equal treatment property.

3 Cooperative games and Gately values

We first discuss the foundational concepts of cooperative games and solution concepts. Let $N = \{1, \ldots, n\}$ be an arbitrary finite set of players and let $2^N = \{S \mid S \subseteq N\}$ be the corresponding set of all (player) coalitions in N. For ease of notation we usually refer to the singleton $\{i\}$ simply as i. Furthermore, we use the simplified notation $S - i = S \setminus \{i\}$ for any $S \in 2^N$ and $i \in S$.

A *cooperative game* on N is a function $v \colon 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$. A game assigns to every coalition a value or "worth" that this coalition can generate through the cooperation of its members. We refer to $v(S)$ as the *worth* of coalition $S \in 2^N$ in the game v. The class of all cooperative games in the player set N is denoted by

$$\mathbb{V}^N = \{v \mid v \colon 2^N \to \mathbb{R} \text{ such that } v(\varnothing) = 0\}.$$

For every player $i \in N$ let $v_i = v(\{i\})$ be her individually feasible worth in the game v. We refer to the game v as being *zero-normalised* if $v_i = 0$ for all $i \in N$. The collection of all zero-normalised games is denoted by $\mathbb{V}_0^N \subset \mathbb{V}^N$.

The set $\mathbb{V}^N$ is a $(2^n - 1)$-dimensional Euclidean vector space. For its analysis it is useful to use the *unanimity basis* of this Euclidean vector space. Here, for every coalition $\varnothing \neq S \subseteq N$ the *S-unanimity game* $u_S \in \mathbb{V}^N$ is defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Every game $v \in \mathbb{V}^N$ can now be written as $v = \sum_{S \neq \varnothing} \Delta_S(v)\, u_S$, where $\Delta_S(v) = \sum_{T \subseteq S} (-1)^{|S|-|T|}\, v(T)$ is the Harsanyi dividend (Harsanyi, 1959) of coalition S in game v.

Marginal contributions and classes of games The *marginal contribution* of an individual player $i \in N$ in the game $v \in \mathbb{V}^N$ is defined by her marginal or "separable" contribution to the grand coalition in this game, i.e., $M_i(v) = v(N) - v(N - i)$ where $N - i = N \setminus \{i\}$. This marginal contribution can be considered as a "utopia value" (Tijs, 1981; Branzei et al., 2008) for the following classes of cooperative games:

Definition 3.1 *A cooperative game $v \in \mathbb{V}^N$ is*

- ***essential*** *if it holds that*

$$\sum_{j \in N} v_j \leqslant v(N) \leqslant \sum_{j \in N} M_j(v) \tag{7}$$

- ***semi-standard*** *if for every player $i \in N$ it holds that*

$$v_i \leqslant M_i(v) \quad \textit{or, equivalently,} \quad v_i + v(N-i) \leqslant v(N) \tag{8}$$

- ***semi-regular*** *if v is essential as well as semi-standard.*

- ***standard*** *if v is semi-standard and, additionally, for at least one player $j \in N$ it holds that $v_j < M_j(v)$, or, equivalently, $v_j + v(N-j) < v(N)$.*

- ***regular*** *if v is essential as well as standard. The collection of regular cooperative games is denoted by $\mathbb{V}^N_\star \subset \mathbb{V}^N$.*[9]

The class of regular cooperative games $\mathbb{V}^N_\star$ is the main domain of analysis for various forms of Gately solutions and their generalisations. In particular, we denote the collection of regular zero-normalised games by $\widehat{\mathbb{V}}^N = \mathbb{V}^N_\star \cap \mathbb{V}^N_0$.

An *allocation* in the game $v \in \mathbb{V}^N$ is any point $x \in \mathbb{R}^N$ such that $x(N) = v(N)$, where we denote by $x(S) = \sum_{j \in S} x_j$ the allocated payoff to the coalition $S \in 2^N$. We denote the class of all allocations for the game $v \in \mathbb{V}^N$ by $\mathbb{A}(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\} \neq \varnothing$. We emphasise that allocations can assign positive as well as negative payoffs to individual players in a game.

An *imputation* in the game $v \in \mathbb{V}^N$ is an allocation $x \in \mathbb{A}(v)$ that is individually rational in the sense that $x_i \geqslant v_i$ for every player $i \in N$. The corresponding imputation set of $v \in \mathbb{V}^N$ is now given by $\mathbb{I}(v) = \{x \in \mathbb{A}(v) \mid x_i \geqslant v_i \text{ for all } i \in N\}$. We remark that for any essential game v with $v(N) > \sum_{i \in N} v_i$ the imputation set $\mathbb{I}(v)$ is a polytope with a non-empty interior. In particular, this holds for the class of regular games $\mathbb{V}^N_\star$.

We recall that for any cooperative game $v \in \mathbb{V}^N$, the *Core* is defined as a set of imputations $C(v) \subset \mathbb{I}(v)$ such that $x \in C(v)$ if and only for all coalitions $S \in 2^N : x(S) \geqslant v(S)$. Hence,

$$C(v) = \{x \in \mathbb{I}(v) \mid x(S) \geqslant v(S) \text{ for all } S \in 2^N\}. \tag{9}$$

Let $\mathbb{V} \subseteq \mathbb{V}^N$ be some collection of TU-games on player set N. A *value* on $\mathbb{V}$ is a map $\phi : \mathbb{V} \to \mathbb{R}^N$ such that $\phi(v) \in \mathbb{A}(v)$ for every $v \in \mathbb{V}$. We emphasise that a value satisfies the efficiency property that $\sum_{i \in N} \phi_i(v) = v(N)$ for every $v \in \mathbb{V}$. We remark that a value ϕ is *individually rational* (IR) if $\phi(v) \in \mathbb{I}(v)$ for all $v \in \mathbb{V}$.

3.1 Gately points and Gately values

As discussed above, Gately's approach is based on a formal notion of the "propensity to disrupt".

[9]We emphasise that every regular game $v \in \mathbb{V}^N$ satisfies a partitional form of superadditivity in the sense that $v(N-i) + v_i \leqslant v(N)$ for every $i \in N$, which is aligned with the notion of a game being *weak constant-sum* as defined in Staudacher and Anwander (2019, Definition 5). Furthermore, Staudacher and Anwander (2019, Theorem 1(a)) is also founded on the class of regular cooperative games.

Definition 3.2 (Gately, 1974; Littlechild and Vaidya, 1976)
Let $v \in \mathbb{V}^N$ be a cooperative game on N. The ***propensity to disrupt*** *of a coalition $S \in 2^N$ at allocation $x \in \mathbb{A}(v)$ with $x(S) \neq v(S)$ is defined by*

$$d(S, x) = \frac{x(N \setminus S) - v(N \setminus S)}{x(S) - v(S)} \tag{10}$$

The ***propensity to disrupt of player*** *$i \in N$ at allocation $x \in \mathbb{A}(v)$ satisfying $x_i \neq v_i$ for every $i \in N$ is given by*

$$d_i(x) = d(\{i\}, x) = \frac{x(N-i) - v(N-i)}{x_i - v_i} = \frac{M_i(v) - x_i}{x_i - v_i} = \frac{M_i(v) - v_i}{x_i - v_i} - 1 \tag{11}$$

A ***Gately point*** *of the game $v \in \mathbb{V}^N$ is defined as an imputation $g \in \mathbb{I}(v)$ that minimises the individual propensities to disrupt, i.e., for all players $i \in N$:*

$$d_i(g) \leqslant \min_{x \in \mathbb{I}(v)} \max_{j \in N} d_j(x) \tag{12}$$

Gately points of cooperative games have most recently been explored by Staudacher and Anwander (2019). They showed the following properties.[10]

Lemma 3.3 (Staudacher and Anwander, 2019)

(a) *Every standard cooperative game $v \in \mathbb{V}^N$ admits a unique Gately point $g(v) \in \mathbb{I}(v)$ given by*

$$g_i(v) = v_i + \frac{M_i(v) - v_i}{\sum_{j \in N} \left(M_j(v) - v_j \right)} \left(v(N) - \sum_{j \in N} v_j \right) \tag{13}$$

for every $i \in N$.

(b) *For every standard zero-normalised game $v \in \mathbb{V}^N$ the unique Gately point is given by*

$$g(v) = \frac{v(N)}{\sum_{j \in N} M_j(v)} M(v) \in \mathbb{I}(v). \tag{14}$$

Lemma 3.3(a) allows us to introduce the **Gately value** as the map $g \colon \mathbb{V}^N_\star \to \mathbb{R}^N$ on the class of regular cooperative games defined by equation (13).

We emphasise that the Gately value is only non-trivially defined on the class of regular cooperative games $\mathbb{V}^N_\star$, while Gately points are in principle defined for arbitrary cooperative games with the property that $M_i(v) \neq v_i$ for some $i \in N$. As pointed out by Staudacher and Anwander (2019), there might be games that admit no Gately points and other games that might admit multiple Gately points.

Gately points and equal treatment Fairness properties have been considered as a main part of the cooperative game theoretic literature. We note that the Gately value as Lemma 3.3(a) clearly satisfies the main fairness properties.

[10]A proof of the properties collected here can be found in Staudacher and Anwander (2019).

A solution f on some class of cooperative games $\mathbb{V}_1^N \subseteq \mathbb{V}^N$ satisfies *equal treatment of equals* if for every $v \in \mathbb{V}_1^N$ and $i, j \in N$ with $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\}$ it holds that $f_i(v) = f_j(v)$. Since $M_i(v) = M_j(v)$ as well as $v_i = v_j$ for the constructed game v, it follows immediately that the Gately value g satisfies the equal treatment property on the class of all standard games.

More generally, we can compute that the following equal treatment property holds for the Gately value. The next statement immediately follows from the definition of the Gately value.

Proposition 3.4 *For every standard cooperative game $v \in \mathbb{V}^N$ and every pair $i, j \in N$:*

$$g_i(v) = g_j(v) \quad \textit{if and only if} \quad \frac{v_i - v_j}{v(N) - \sum_h v_h} + \frac{M_i(v) - M_j(v)}{v(N) - \sum_h M_h(v)} = 0.$$

Generalised Gately values We generalise the notion of Gately points. The next definition introduces a generalised notion of the Gately value on the class of standard cooperative games.

Definition 3.5 *Let $v \in \mathbb{V}^N$ be some standard cooperative game on N. For any parameter value $\alpha > 0$ we define the α-**Gately value** of game v as the imputation $g^\alpha(v) \in \mathbb{I}(v)$ with*

$$g_i^\alpha(v) = v_i + \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N} \left(M_j(v) - v_j\right)^\alpha} \left(v(N) - \sum_{j \in N} v_j \right) \qquad \textit{for every } i \in N. \tag{15}$$

*We refer to $\mathcal{G} = \{g^\alpha \colon \mathbb{V}_\star^N \to \mathbb{R}^N \mid \alpha > 0\}$ as the family of generalised Gately values on the domain of regular cooperative games $\mathbb{V}_\star^N$. For any $v \in \mathbb{V}_\star^N$ the related set $\mathcal{G}(v) = \{g^\alpha(v) \mid \alpha > 0\} \subseteq \mathbb{I}(v)$ defines the **Gately set** for that particular game.*

From this definition we can identify some special cases:

- We note that $g^1 = g \in \mathcal{G}$ is the original Gately value on the class of regular games $\mathbb{V}_\star^N$.
- Although g^α is not defined for $\alpha = 0$, note that

  $$\lim_{\alpha \downarrow 0} g_i^\alpha(v) = v_i + \tfrac{1}{|N_0(v)|} \left(v(N) - \sum_{j \in N} v_j \right)$$

  for all $i \in N_0(v)$ and $\lim_{\alpha \downarrow 0} g_i^\alpha(v) = v_i$ for $i \in N \setminus N_0(v)$, where $N_0(v) = \{i \in N \mid M_i(v) > v_i\} \neq \varnothing$. This compares to the CIS value of v (Driessen and Funaki, 1991).
  Furthermore, if the game $v \in \widehat{\mathbb{V}}^N$ is additionally zero-normalised, $\lim_{\alpha \downarrow 0} g_i^\alpha(v)$ corresponds to the equal division value given by $E(v) = \frac{v(N)}{n}$ if $M_i(v) \neq 0$ for all $i \in N$.
- Finally, $\lim_{\alpha \to \infty} g_i^\alpha(v) = v_i + \frac{1}{|N_1(v)|} \left(v(N) - \sum_{j \in N} v_j \right)$ for all $i \in N_1(v)$ and $\lim_{\alpha \to \infty} g_i^\alpha(v) = v_i$ for $i \in N \setminus N_1(v)$, where $N_1(v) = \{i \in N \mid M_i(v) - v_i = \max_{j \in N}(M_j(v) - v_j) \} \neq \varnothing$. Again, this can be interpreted as a variation of the CIS value.

The next definition introduces a generalised formulation of Gately's seminal notion of the propensity to disrupt. We show that α-Gately values are closely related to optimisation problems based on this generalised notion.

Definition 3.6 *Let $v \in \mathbb{V}^N$ be some cooperative game on player set N. For every parameter $\beta > 0$ the corresponding* ***generalised β-propensity to disrupt of player*** *$i \in N$ at imputation $x \in \mathbb{I}(v)$ with $x_i \neq v_i$ is defined by*

$$\rho_i^\beta(x) = \frac{M_i(v) - v_i}{(x_i - v_i)^\beta} \tag{16}$$

We note here that for $\beta = 1$, this generalised propensity to disrupt corresponds exactly to the original propensity to disrupt for an individual player as introduced by Gately (1974), in the sense that $\rho_i^1(x) = \frac{M_i(v)-v_i}{x_i-v_i} = d_i(x) + 1$.

The following theorem shows the relationship between the balancing of such generalised propensities to disrupt and corresponding α-Gately values. In particular, it is shown that the α-Gately value can be interpreted as a bargaining value, like the original Gately value and the Nucleolus. Furthermore, for certain values of α, the minimisation of the generated total generalised propensity to disrupt at an allocation results in the corresponding α-Gately value.

Theorem 3.7 *Let $v \in \mathbb{V}^N_\star$ be a regular cooperative game on N.*

(a) *Let $\alpha > 0$ and define $\beta = \frac{1}{\alpha}$. Then the α-Gately value $g^\alpha(v) \in \mathbb{I}(v)$ is the unique β-Gately point in the sense that $g^\alpha(v)$ is the unique imputation that satisfies the property that*

$$\rho_i^\beta\left(g^\alpha(v)\right) \leqslant \min_{x \in \mathbb{I}(v)} \max_{j \in N} \rho_j^\beta(x) \tag{17}$$

for every player $i \in N$.

(b) *Let $0 < \alpha < 1$ and define $\beta = \frac{1-\alpha}{\alpha} > 0$. Then the α-Gately value $g^\alpha(v) \in \mathbb{I}(v)$ is the unique solution to the minimisation of the total aggregated generalised β-propensity to disrupt of the game v:*

$$g^\alpha(v) = \arg\min_{x \in \mathbb{I}(v)} \sum_{j \in N} \rho_j^\beta(x) \tag{18}$$

Proof. To show assertion (a), we note that for every imputation $x \in \mathbb{I}(v)$ and every player $i \in N$ with $x_i \neq v_i$: $\rho_j^\beta(x) = \frac{M_i(v)-v_i}{(x_i-v_i)^\beta} \geqslant 0$ from the hypothesis that $M_i(v) \geqslant v_i$. Furthermore, since $M_j(v) > v_j$ for at least one $j \in N$, we conclude that $r = \max_{j \in N} \rho_j^\beta(x) > 0$.
Following the method developed by Staudacher and Anwander (2019), the defining minimax problem $\min_{x \in \mathbb{I}(v)} \max_{j \in N} \rho_j^\beta(x)$ can be solved by identifying $r > 0$ and an imputation $x^* \in \mathbb{I}(v)$ such that $\rho_i^\beta(x^*) = r$ for all $i \in N_0 = \{i \in N \mid M_i(v) > v_i\} \neq \varnothing$.
First, note that for all $j \in N \setminus N_0 = \{j \in N \mid M_j(v) = v_j\}$ we can set $x_j = v_j$. Next, for $i \in N_0$ we can now solve for $r > 0$ as well as x_i. Rewriting $\rho_i^\beta(x) = r$, we derive that

$$x_i = \left(\frac{M_i(v) - v_i}{r}\right)^{\frac{1}{\beta}} + v_i.$$

Note that, since $M_i(v) \geqslant v_i$ for all $i \in N$, it follows that $x_i \geqslant 0$ for every $i \in N$. It must be that

$x \in \mathbb{I}(v)$. Hence,

$$\sum_{i \in N} x_i = \frac{\sum_{i \in N_0} (M_i(v) - v_i)^{\frac{1}{\beta}}}{r^{\frac{1}{\beta}}} + \sum_{i \in N} v_i \equiv v(N).$$

Since $\sum_{i \in N_0} (M_i(v) - v_i)^{\frac{1}{\beta}} = \sum_{i \in N} (M_i(v) - v_i)^{\frac{1}{\beta}}$, we conclude that

$$r = \frac{\left[\sum_{i \in N} (M_i(v) - v_i)^{\frac{1}{\beta}} \right]^{\beta}}{\left[v(N) - \sum_{j \in N} v_j \right]^{\beta}} > 0.$$

From this we conclude that the identified solution exists and is unique under the regularity conditions on the game v.

Substituting the formulated solution of r back into the formulation for the solution, we deduce that

$$x_i = v_i + \frac{(M_i(v) - v_i)^{\frac{1}{\beta}}}{\sum_{j \in N} (M_j(v) - v_j)^{\frac{1}{\beta}}} \left(v(N) - \sum_{j \in N} v_j \right) \geqslant v_i.$$

Recalling that $\beta = \frac{1}{\alpha}$, we indeed conclude that $x_i = g_i^{\alpha}(v)$, leading us to conclude that assertion (a) has been shown.

To show assertion (b), consider the minimisation problem $\min_{x \in \mathbb{I}(v)} R^{\beta}(x)$ as formulated, where $R^{\beta}(x) = \sum_{j \in N} \rho_j^{\beta}(x)$. Deriving the Lagrangian $L(x_1, ..., x_n, \lambda) = \sum_{i \in N} \left[\frac{M_i(v) - v_i}{(x_i - v_i)^{\beta}} \right] + \lambda(\sum_{i \in N} x_i - v(N))$, and deriving the necessary first-order conditions, we conclude that

$$\frac{M_1(v) - v_1}{(x_1 - v_1)^{\beta+1}} = \frac{M_2(v) - v_2}{(x_2 - v_2)^{\beta+1}} = \cdots = \frac{M_n(v) - v_n}{(x_n - v_n)^{\beta+1}}.$$

Thus, we arrive at $n - 1$ equations given by

$$x_k - v_k = \frac{(M_2(v) - v_2)^{\frac{1}{\beta+1}}}{(M_1(v) - v_1)^{\frac{1}{\beta+1}}} (x_1 - v_1) \qquad \text{for } k = 2, \ldots, n.$$

This we can rewrite as

$$v(N) - \sum_{j=3}^{n} x_j - x_1 - v_2 = \frac{(M_2(v) - v_2)^{\frac{1}{\beta+1}}}{(M_1(v) - v_1)^{\frac{1}{\beta+1}}} (x_1 - v_1)$$

together with

$$x_k = v_k + \frac{(M_k(v) - v_k)^{\frac{1}{\beta+1}}}{(M_1(v) - v_1)^{\frac{1}{\beta+1}}} (x_1 - v_1) \qquad \text{for } k = 3, \ldots, n.$$

Summing up the LHSs and the RHSs, we have the following equalities:

$$v(N) - x_1 - v_2 = v_3 + \ldots + v_n + \sum_{j=2}^{n} \frac{(M_j(v) - v_j)^{\frac{1}{\beta+1}}}{(M_1(v) - v_1)^{\frac{1}{\beta+1}}}(x_1 - v_1)$$

This leads to the conclusion that

$$\begin{aligned} v(N) - \sum_{j=2}^{n} v_j &= x_1 + \frac{(x_1 - v_1)}{(M_1(v) - v_1)^{\frac{1}{\beta+1}}} \sum_{j=2}^{n} (M_j(v) - v_j)^{\frac{1}{\beta+1}} \\ &= x_1 + \frac{(x_1 - v_1)}{(M_1(v) - v_1)^{\frac{1}{\beta+1}}} \sum_{j=1}^{n} (M_j(v) - v_j)^{\frac{1}{\beta+1}} - (x_1 - v_1) \end{aligned}$$

Hence, we conclude that

$$\frac{(M_1(v) - v_1)^{\frac{1}{\beta+1}}}{\sum_{j=1}^{n} (M_j(v) - v_j)^{\frac{1}{\beta+1}}} [v(N) - \sum_{j=1}^{n} v_j] = x_1 - v_1.$$

Remarking that $\alpha = \frac{1}{\beta+1}$ leads us immediately to the insight that the first player's allocation is actually her α-Gately value value. The resulting allocations for the other players $j = 2, \ldots, n$ are derived in a similar fashion. ■

We remark that Theorem 3.7 applies to regular cost games or problems as well. Indeed, for a *cost game* $v \in \mathbb{V}^N$ satisfying $v(N) \leqslant \sum_{j \in N} v_J$, $M_i(v) \leqslant v_i \leqslant 0$ for all $i \in N$ and $M_j(v) < v_j \leqslant 0$ for some $j \in N$, both assertions of Theorem 3.7 hold. We do not consider these games here, but refer to, e.g., Moulin (2004) for a discussion of these cost games.

To illustrate the importance of regularity of those cooperative games for which Gately values are well-defined as imposed in Theorem 3.7(a), we consider the next example of a three-player game that exhibits non-regularities.

Example 3.8 Consider a 3-player game v on $N = \{1, 2, 3\}$ defined by $v_1 = 2$, $v_2 = 1$, $v_3 = 0$, $v(12) = v(13) = v(23) = 4$ and $v(N) = 5$.
We remark that the marginal contributions are $M_1(v) = M_2(v) = M_3(v) = 1$, leading to the conclusion that $M_1(v) - v_1 = -1 < 0$, $M_2(v) = v_2$, and $M_3(v) - v_3 = 1 > 0$. Hence, this game is neither essential nor semi-standard.[11]
It is straightforward to establish that this game admits a continuum of Gately points identified as $\{(t, 3, 2 - t) \mid 0 \leqslant t \leqslant 2\} \subset \mathbb{A}(v)$. ♦

With regard to Theorem 3.7(b) we remark that for $\beta = 1$ the minimisation of the aggregated total propensity to disrupt $R^1(x) = \sum_{i \in N} d_i(x)$ results in the $\frac{1}{2}$-Gately value as the unique solution. Furthermore, if $\beta = 0$, the generalised propensity to disrupt for any player $i \in N$ is no longer a function of the allocation x_i, implying that the total aggregated 0-propensity to disrupt is a constant function. This implies that the minimisation problem (18) has a continuum of solutions, including all Gately points.

[11]We also remark that this game has actually an empty Core.

3.2 Dual Gately values

Let $v \in \mathbb{V}^N$ be a cooperative game. Then the *dual game* of v, denoted by $v^* \colon 2^N \to \mathbb{R}$, is defined by

$$v^*(S) = v(N) - v(N \setminus S) \qquad \text{for every coalition } S \in 2^N \tag{19}$$

The dual of a game assigns to every coalition $S \subseteq N$ the worth that is lost by the grand coalition N if coalition S leaves the game. Note in particular that $v^*(\varnothing) = 0$, $v^*(N) = v(N)$ and $v_i^* = v^*(\{i\}) = v(N) - v(N - i) = M_i(v)$ for all $i \in N$. Finally, $M_i(v^*) = v_i$ for every $i \in N$.

We investigate the "dual" of a given value, which assign to games the value of its dual game. As an illustrative example, we note that Driessen and Funaki (1991) considered the dual of the CIS-value, defined as the CIS-value of the dual game. They refer to this notion as the "Egalitarian Non-Separable Contribution" value, or ENSC-value.

We can apply a similar procedure to the Gately value. We note first that the dual of a Gately value only can properly formulated for parameter values that are natural numbers, i.e., $\alpha \in \mathbb{N}$. This is subject to the next definition.

Definition 3.9 *Let $\alpha \in \mathbb{N}$. The **dual α-Gately value** is a map $\overline{g^\alpha} \colon \mathbb{V}^N_\star \to \mathbb{R}^N$ that assigns to every regular cooperative game $v \in \mathbb{V}^N_\star$ the α-Gately value of its dual game $v^* \in \mathbb{V}^N$, i.e., $\overline{g^\alpha}(v) = g^\alpha(v^*) \in \mathbb{A}(v)$.*

The next proposition considers some properties of dual α-Gately values.

Proposition 3.10 *Consider a regular cooperative game $v \in \mathbb{V}^N_\star$ and let $\alpha \in \mathbb{N}$ be a natural number. Then the following properties hold:*

(a) *For every $\alpha \in \mathbb{N}$ the dual α-Gately value of v is well-defined and given by*

$$\overline{g_i^\alpha}(v) = M_i(v) - \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} \left(\sum_{j \in N} M_j(v) - v(N) \right) \tag{20}$$

for every player $i \in N$.

(b) *The dual α-Gately value of v is identical to the α-Gately value of v, i.e., $\overline{g^\alpha}(v) = g^\alpha(v)$, if and only if $\alpha = 1$ or $M_i(v) - v_i = M_j(v) - v_j \geqslant 0$ for all $i, j \in N$.*

Proof. To show assertion (a), let $\alpha \in \mathbb{N}$. We compute that for every player $i \in N$:

$$\begin{aligned} \overline{g_i^\alpha}(v) = g_i^\alpha(v^*) &= v_i^* + \frac{(M_i(v^*) - v_i^*)^\alpha}{\sum_{j \in N} (M_j(v^*) - v_j^*)^\alpha} \left(v^*(N) - \sum_{j \in N} v_j^* \right) \\ &= M_i(v) - \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} \left(\sum_{j \in N} M_j(v) - v(N) \right) \end{aligned}$$

with the remark that $(-1)^\alpha$ attains only the values 1 and -1 due to $\alpha \in \mathbb{N}$.
Furthermore, we note that $\sum_{i \in N} \overline{g_i^\alpha}(v) = v(N)$, thereby showing that the dual α-Gately value is indeed well-defined.

To show assertion (b) let $i \in N$ and $\alpha \in \mathbb{N}$. We now note that from assertion (a) $g_i^\alpha(v) = \overline{g_i^\alpha}(v)$ if and only if

$$v_i + \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N}(M_j(v) - v_j)^\alpha} \left(v(N) - \sum_{j \in N} v_j \right) = M_i(v) - \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N}(M_j(v) - v_j)^\alpha} \left(\sum_{j \in N} M_j(v) - v(N) \right)$$

or

$$\frac{\sum_{i \in N} (M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)} = (M_i(v) - v_i)^{\alpha - 1}$$

This is valid for all $i \in N$ if and only if $\alpha = 1$ or $M_i(v) - v_i = M_j(v) - v_j \geqslant 0$ for all $i, j \in N$. ■

Proposition 3.10 (b) implies immediately that the dual Gately value is the same as the Gately value on the class of regular games. This is stated in the next corollary.

Corollary 3.11 *For every regular cooperative game $v \in \mathbb{V}_\star^N$, the dual Gately value of v is identical to the Gately value of v, i.e., $\overline{g}_i(v) = g_i(v)$ for all $i \in N$.*

3.3 Characterisations of the Gately value

It is easy to see that the Gately value on the class of regular games $\mathbb{V}_\star^N$ is a compromise value of the individual worth vector and the net marginal contribution vector. Indeed, for any regular game $v \in \mathbb{V}_\star^N$, the individual worth vector is given by $\nu_v = (v_1, \dots, v_n)$ and the net marginal contribution vector by $\bar{n} = M(v) - \nu_v = (M_1(v) - v_1, \dots, M_n(v) - v_n)$. Now the Gately value $g(v)$ for game v can be written as

$$g(v) = (1 - \gamma_v)\nu_v + \gamma_v M(v) \qquad \text{where } \gamma_v = \frac{v(N) - \sum_{i \in N} v_i}{\sum_{i \in N}(M_i(v) - v_i)}. \tag{21}$$

This re-interpretation of the Gately value as a compromise value holds on the class of regular games $\mathbb{V}_\star^N$ and allows us to characterise the Gately value as such a compromise value.

We remark that the compromise values form a distinct subclass of solution concepts that are based on a similar methodology of determining the exact allocation of the worth of the grand coalition N. This refers to the fundamental property that the value is a convex combination of a well-defined lower and upper bound such that the value satisfies efficiency. We refer to Tijs and Otten (1993) and Gilles and van den Brink (2023) for detailed discussion and characterisations of these solution concepts.

An axiomatisation of the Gately value Tijs (1987) devised a simple axiomatisation for the τ-value (Tijs, 1981) that is completely based on the property that the τ-value is a compromise value. We can devise a similar axiomatisation of the Gately value by replicating Tijs's characterisation methodology to the Gately value to arrive at the following axiomatisation.

In this characterisation, a variant of the compromise property and the restricted proportionality property, seminally introduced by Tijs (1987) on the class of quasi-balanced games, can be constructed on the class of regular cooperative games $\mathbb{V}_\star^N$.

Theorem 3.12 *The Gately value g is the unique map $f\colon \mathbb{V}^N_\star \to \mathbb{R}^N$ on the class of regular games $\mathbb{V}^N_\star$ that satisfies the following three properties:*

(i) ***Efficiency:*** *$\sum_{i\in N} f_i(v) = v(N)$ for every $v \in \mathbb{V}^N_\star$;*

(ii) *ν_v-**Compromise property:** For every regular game $v \in \mathbb{V}^N_\star\colon f(v) = \nu_v + f(v - \nu_v)$, where $v - \nu_v \in \widehat{\mathbb{V}}^N$ is the zero-normalisation of v defined by $(v - \nu_v)(S) = v(S) - \sum_{i\in S} v_i$ for every coalition $S \in 2^N$, and;*

(iii) ***Restricted proportionality property:** For every zero-normalised regular cooperative game $v \in \widehat{\mathbb{V}}^N\colon f(v) = \gamma_v M(v)$ for some $\gamma_v \in \mathbb{R}$.*

Proof. We first show that the Gately value $g\colon \mathbb{V}^N_\star \to \mathbb{R}^N$ satisfies the three stated properties. For that purpose let $v \in \mathbb{V}^N_\star$.

(i) Obviously the Gately value $g(v)$ is efficient for v.

(ii) Let $w = v - \nu_v \in \mathbb{V}^N_0$ be the zero-normalisation of v. Then for every $i \in N$ we deduce that $w_i = v_i - v_i = 0$ and $M_i(w) = w(N) - w(N-i) = v(N) - v(N-i) - v_i = M_i(v) - v_i$. Hence, $M_i(w) \geqslant 0 = w_i$ and for those players $j \in N$ with $M_j(v) > v_j$ we deduce that $M_j(w) > 0 = w_j$.
Furthermore, $\sum_{i\in N} v_i \leqslant v(N) \leqslant \sum_{i\in N} M_i(v)$ is equivalent to $0 \leqslant v(N) - \sum_{i\in N} v_i = w(N) \leqslant \sum_{i\in N} M_i(v) - \sum_{i\in N} v_i = \sum_{i\in N} M_i(w)$, implying that $w \in \mathbb{V}^N_\star$. Therefore, $w = v - \nu_v \in \widehat{\mathbb{V}}^N$.
Now by definition for every $i \in N$:

$$g_i(w) = \frac{M_i(w)}{\sum_{j\in N} M_j(w)} \cdot w(N) = \frac{M_i(v) - v_i}{\sum_{j\in N}(M_j(v) - v_j)} \cdot \left(v(N) - \sum_{j\in N} v_j \right) = g_i(v) - v_i.$$

This shows that $g_i(v) = v_i + g_i(v - \nu_v)$.

(iii) Assume that $v \in \widehat{\mathbb{V}}^N$. Then for any $i \in N\colon g_i(v) = \frac{M_i(v)}{\sum_{j\in N} M_j(v)} \cdot v(N)$ showing restricted proportionality with $\gamma_v = \frac{v(N)}{\sum_{j\in N} M_j(v)}$.

Next, we show that if $f\colon \mathbb{V}^N_\star \to \mathbb{R}^N$ satisfies the three stated properties, it is equal to the Gately value. Take any regular game $v \in \mathbb{V}^N_\star$ and let $w = v - \nu_v \in \widehat{\mathbb{V}}^N$ be its zero-normalisation.
Then from restricted proportionality we have that $f(w) = \gamma_w M(w) = \gamma_w\,(M(v) - \nu_v)$. Furthermore, from the compromise property we conclude that

$$f(v) = \nu_v + f(v - \nu_v) = \nu_v + \gamma_v(M(v) - \nu_v).$$

Using efficiency we then conclude that

$$\sum_{i\in N} f_i(v) = \sum_{i\in N} v_i + \gamma_v \left(\sum_{i\in N} M_i(v) - \sum_{i\in N} v_i \right) = v(N)$$

implying that

$$\gamma_v = \frac{v(N) - \sum_{i\in N} v_i}{\sum_{i\in N}(M_i(v) - v_i)}.$$

We immediately conclude from this that $f_i(v) = g_i(v)$, showing the assertion. ■

We note that the three properties in this axiomatisation are independent:

- The *efficiency* property is a well-established property that is used throughout the literature. It guarantees that the allocation rule selects from the set of imputations in the game rather than the broader set of allocations. We note that the allocation rule $f(v) = M(v)$ on $\mathbb{V}^N_\star$ clearly satisfies the compromise property as well as the restricted proportionality property, but which is not efficient.

- The v_v*-compromise property* is a reduced form of additivity and as such decomposes the allocation rule in a translation of the allocation assigned to the zero-normalisation of the game. This property originated in Tijs (1987) as the "compromise property" for the minimal right vector $m(v)$ rather than the vector of individual worths v_v. It is clear that the τ-value satisfies efficiency and the restricted proportionality property. It does not satisfy the v_v-compromise property, but rather the compromise property based on the minimal rights vector $m(v)$.

- The *restricted proportionality property* imposes zero-normalised games are assigned an allocation that is proportional to the utopia vector $M(v)$. This property originates from Tijs (1987) as well and it is satisfied by the τ-value. On the other hand, the Shapley value is a solution concept that is efficient and satisfies the v_v-compromise property, but it does not satisfy restricted proportionality.

4 Gately points and the Core for 3-player games

Gately (1974) introduced his solution concept as a Core selector within the setting of three-player games only, even though Gately did not investigate the exact conditions under which this solution is indeed in the Core. Littlechild and Vaidya (1976) point out that Gately's conception does not necessarily result in a Core selector for games with more than three players, devising a counterexample for 4 players.

In this section we first discuss the relationship between the Gately value and the Core for games with three players only. This is an exceptional case, since the worths of all coalitions in a three-player game are featured in the computation of the Gately value, in contrast to games with more than three players, in which worths of medium-sized coalitions are not considered. This is further explored in the second part of this section, which considers the relationship between the Gately value and the Core of cooperative games with an arbitrary number of players.

We are able to confirm that there is a strong relationship between Gately points and the Core in three-player games. We first illustrate that there exist essential games with empty Cores for which the unique Gately point is well-defined.

Example 4.1 Consider an essential three-player game with $N = \{1, 2, 3\}$ and v given by $v_1 = 5$, $v_2 = v_3 = 0$, $v(12) = v(13) = 1$, $v(23) = 5$ and $v(N) = 6$.
First note that v is indeed essential, since $M_1(v) = 1$ and $M_2(v) = M_3(v) = 5$. On the other hand, v is

not semi-standard, since $v_1 = 5 > M_1(v) = 1$.
Note that the Core of this game is empty, since for an allocation $x \in \mathbb{A}(v)$ with $x(N) = v(N) = 6$ and $x_2 + x_3 \geqslant v(23) = 5$ it follows that $x_1 \leqslant 1$. This is contradiction to the Core requirement that $x_1 \geqslant v_1 = 5$.
Regarding the existence of Gately points for this particular game, we note that the minimax optimisation problem can be re-stated here as the balance equation $\frac{M_1(v)-v_1}{x_1-v_1} = \frac{M_2(v)-v_2}{x_2-v_2} = \frac{M_3(v)-v_3}{x_3-v_3}$ resulting into $\frac{-4}{x_1-5} = \frac{5}{x_2} = \frac{5}{x_3}$, which leads to a unique Gately point $g_1 = 4\frac{1}{3}$ and $g_2 = g_3 = \frac{5}{6}$. Note that this unique Gately point can also be computed by the Gately value formula stated in equation (13).
In comparison, the Shapley value of this game is given by $\phi = \left(2\frac{1}{3}, 1\frac{5}{6}, 1\frac{5}{6}\right)$. ♦

The next theorem gathers some properties of three-player games regarding the relationship between the Core and the Gately points of these games. These properties generalise the insights presented through the previous two examples.

Theorem 4.2 *Let $v \in \mathbb{V}^N$ be a three-player game on $N = \{1, 2, 3\}$. Then the following properties hold:*

(a) *If the game v is semi-regular, then the Gately value is in its Core, $g(v) \in C(v) \neq \varnothing$.*

(b) *If $C(v) \neq \varnothing$, then the game v is semi-regular and $g(v) \in C(v)$.*

Proof. To show assertion (a), we first consider a three-player game $v \in \mathbb{V}^N$ that is semi-regular, but not regular. Hence, $M_i(v) = v_i$ for $i = 1, 2, 3$, implying that $v_1+v_2+v_3 = M_1(v)+M_2(v)+M_3(v) = v(N)$. Simple computations show that there is a unique Core imputation given by $C(v) = \{\, (v_1, v_2, v_3) \,\} = \{\, (\, M_1(v), M_2(v), M_3(v) \,) \,\} \neq \varnothing$. Furthermore, it is easily established that the unique Gately point is well-defined and given by $g(v) = (v_1, v_2, v_3) \in C(v)$.

Next, we assume that v is regular in the sense that $v_1 + v_2 + v_3 \leqslant v(N) \leqslant M_1(v) + M_2(v) + M_3(v)$, $v_i \leqslant M_i(v)$ for all $i = 1, 2, 3$ and, without loss of generality, $v_1 < M_1(v)$. Hence, it holds that $v(12)+v(13)+v(23) \leqslant 2v(N)$. Furthermore, it follows that $3v(N)-v(12)-v(13)-v(23)-v_1-v_2-v_3 = \sum_j \left(M_j(v) - v_j\right) > 0$.
Now define for every $i = 1, 2, 3$

$$\eta_i = \frac{2v(N) - v(12) - v(13) - v(23)}{3v(N) - v(12) - v(13) - v(23) - v_1 - v_2 - v_3} \; (M_i(v) - v_i\,)$$

Note that $\eta_i \geqslant 0$ for all $i = 1, 2, 3$ and that, in particular, $\eta_1 > 0$.
We now note the following properties of these introduced quantities:

- First, regarding their sum,

$$\begin{aligned}\eta_1 + \eta_2 + \eta_3 &= \frac{2v(N) - v(12) - v(13) - v(23)}{3v(N) - v(12) - v(13) - v(23) - \sum_i v_i} \sum_i (M_i(v) - v_i\,) \\ &= \sum_i M_i(v) - v(N) = 2v(N) - v(12) - v(13) - v(23).\end{aligned}$$

- Second, for every $i = 1, 2, 3$:

$$\eta_i = \frac{\sum_j M_j(v) - v(N)}{\sum_j \big(M_j(v) - v_j\big)} \, (M_i(v) - v_i) \leqslant M_i(v) - v_i.$$

- Finally, for every $i = 1, 2, 3$ we argue that $g_i(v) = M_i(v) - \eta_i$. Indeed,

$$\begin{aligned} g_i(v) &= \frac{M_i(v) - v_i}{\sum_j \big(M_j(v) - v_j\big)} \left(v(N) - \sum_j v_j \right) \\ &= M_i(v) + \frac{\sum_j M_j(v) - v(N)}{\sum_j \big(M_j(v) - v_j\big)} \, (v_i - M_i(v)) = M_i(v) - \eta_i. \end{aligned}$$

Using the argument of Vorob'ev (1977, 4.12.1), we now claim that $g(v) \in C(v)$ using the construction above. We check the conditions for $g(v)$ being a Core selector:

First, for every $i = 1, 2, 3$: $g_i(v) = M_i(v) - \eta_i \geqslant M_i(v) - (M_i(v) - v_i) = v_i$.

Second, we can check for each 2-player coalition the Core conditions. For $\{1, 2\}$ it is easy to see that

$$\begin{aligned} g_1(v) + g_2(v) &= M_1(v) + M_2(v) - \eta_1 - \eta_2 \\ &= 2v(N) - v(23) - v(13) - \eta_1 - \eta_2 \\ &= v(12) + \eta_3 \geqslant v(12) \end{aligned}$$

Similar arguments show that $g_1(v) + g_3(v) \geqslant v(13)$ and $g_2(v) + g_3(v) \geqslant v(23)$.

Together with $g_1(v) + g_2(v) + g_3(v) = v(N)$, this completes the proof of assertion (a).

To show assertion (b), assume that for three-player game $v \in \mathbb{V}^N$ with $N = \{1, 2, 3\}$ it holds that $C(v) \neq \varnothing$. Hence, there exists some $(x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_1 + x_2 + x_3 = v(N)$, $x_i \geqslant v_i$ for $i = 1, 2, 3$, and $x_1 + x_2 \geqslant v(12)$, $x_1 + x_3 \geqslant v(13)$, and $x_2 + x_3 \geqslant v(23)$.

Adding the last three inequalities results in the conclusion that

$$2v(N) = 2x_1 + 2x_2 + 2x_3 \geqslant v(12) + v(13) + v(23),$$

which in turn leads to the conclusion that

$$M_1(v) + M_2(v) + M_3(v) = (v(N) - v(12)) + (v(N) - v(13)) + (v(N) - v(23)) \geqslant v(N).$$

Furthermore, from $x_i \geqslant v_i$ for $i = 1, 2, 3$ it follows that $v(N) = x_1 + x_2 + x_3 \geqslant v_1 + v_2 + v_3$.

These two inequalities leads us to the conclusion that $v_1 + v_2 + v_3 \leqslant v(N) \leqslant M_1(v) + M_2(v) + M_3(v)$, implying that v is indeed essential.

Furthermore, $v(N) = x_1 + (x_2 + x_3) \geqslant v_1 + v(23)$ implying that $M_1(v) = v(N) - v(23) \geqslant v_1$. This argument can be replicated for players 2 and 3, leading to the desired conclusion that v is indeed semi-regular. The conclusion that, therefore, $g(v) \in C(v)$ follows from assertion (a). ∎

An immediate insight from Theorem 4.2 is that for every three-player game with a non-empty Core, the Gately value is a Core selector:

Corollary 4.3 *Let $v \in \mathbb{V}^N$ with $N = \{1, 2, 3\}$ be a three-player cooperative game. Then $g(v) \in C(v)$ if and only if $C(v) \neq \varnothing$.*

Similar arguments as the ones used in the proof of Theorem 4.2(b) show that the Gately value of certain semi-regular three-player games is equal to the vector of marginal contributions and it is the unique Core imputation if the Core is non-empty.

Corollary 4.4 *Let $v \in \mathbb{V}^N$ with $N = \{1, 2, 3\}$ be a three-player cooperative game such that v is an essential cooperative game such that $v(12) + v(13) + v(23) = 2v(N)$. Then the unique Gately point coincides with the Nucleolus of the game, being equal to the vector of marginal contributions $g(v) = \mathcal{N}(v) = (\, M_1(v), M_2(v), M_3(v)\,)$.*
Furthermore, if $C(v) \neq \varnothing$, it holds that $C(v) = \{g(v)\} = \{\mathcal{N}(v)\}$.

The conclusions about the equivalence of the Gately value and the Nucleolus of the three-player game in case that $C(v) = \varnothing$, stated in Corollary 4.4, follows from application of Leng and Parlar (2010, Theorem 1).

α-Gately values and the Core of 3-player games The analysis of the relationship between α-Gately values and the Core of a three-player game is more complex if we look beyond the standard Gately value ($\alpha = 1$).

The next example shows that there exist three-player games in which α-Gately values are in the Core for a certain closed interval of α values bounded away from zero.

Example 4.5 Consider a zero-normalised three-player game v with $N = \{1, 2, 3\}$ and $v_i = 0$ for $i = 1, 2, 3$, $v(12) = 12$, $v(13) = v(23) = 7$ and $v(N) = 16$. Clearly, this game is regular.
We easily compute that the marginal contributions are given by $M_1(v) = M_2(v) = 9$ and $M_3(v) = 4$. For any $\alpha > 0$ we compute the α-Gately values as

$$g_1^\alpha(v) = g_2^\alpha(v) = \frac{16 \cdot 9^\alpha}{2 \cdot 9^\alpha + 4^\alpha} \text{ and } g_3^\alpha(v) = \frac{16 \cdot 4^\alpha}{2 \cdot 9^\alpha + 4^\alpha}$$

We note that there are essentially two characteristic inequalities to determine whether the α-Gately value in the Core of v:

$$g_1^\alpha(v) + g_2^\alpha(v) \geqslant v(12) = 12 \;\text{ and }\; g_1^\alpha(v) + g_3^\alpha(v) = g_2^\alpha(v) + g_3^\alpha(v) \geqslant v(13) = v(23) = 7$$

Therefore, the range of α values for which the α-Gately value is in the Core of this game is given by $\alpha \in A^* = \left[\, \frac{\ln 3 - \ln 2}{\ln 9 - \ln 4}, \infty \right) = \left[\, \frac{1}{2}, \infty \,\right)$. Note $1 \in A^*$, implying that $g(v) \in C(v)$.
The Shapley value of this game is computed as $\phi = \left(6\frac{1}{6}, 6\frac{1}{6}, 3\frac{2}{3}\right) = g^{\alpha^\bullet}(v) \in C(v)$ with $\alpha^\bullet = \frac{\ln 37 - \ln 22}{\ln 9 - \ln 4} \approx 0.6411$. ♦

The next example considers a game with a large set of imputations and a minimal Core, consisting of a single imputation. For this example we show that the original Gately value is the only Core selector, while all α-Gately values for $\alpha \neq 1$ are outside the Core.

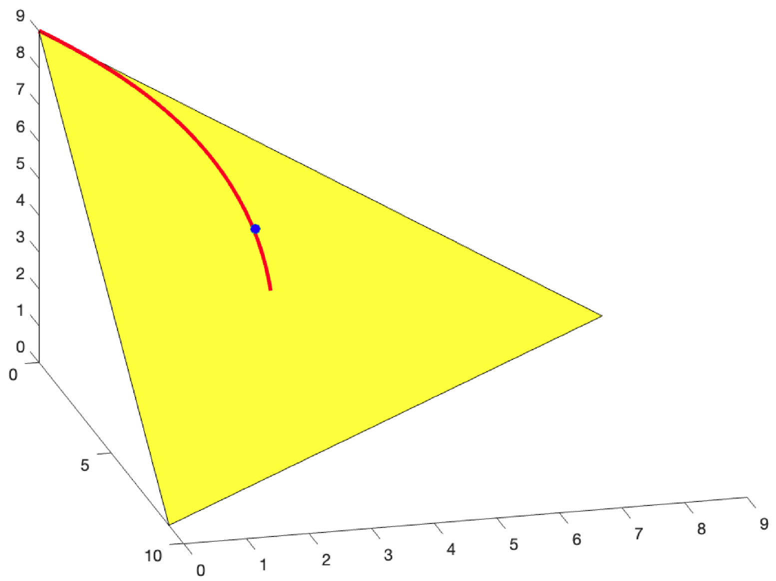

Figure 1: The Core and the trajectory of α-Gately values in Example 4.6.

Example 4.6 Consider a regular three-player game with $N = \{1, 2, 3\}$ and v given by $v_1 = v_2 = v_3 = 0$, $v(12) = 5$, $v(13) = 6$, $v(23) = 7$ and $v(N) = 9$. Note that $\mathbb{I}(v) = \{x \in \mathbb{R}^3_+ \mid \sum x_i = 9\}$ and that the Core is a singleton with $C(v) = \{\,(2, 3, 4)\,\} = \{\,M(v)\,\}$.
We note that for this game $g(v) = M(v)$ selects the unique Core imputation. However, for all $\alpha > 0$ with $\alpha \neq 1$ we have that

$$g^\alpha(v) = \frac{9}{2^\alpha + 3^\alpha + 4^\alpha}\,(\,2^\alpha, 3^\alpha, 4^\alpha\,) \neq (2, 3, 4).$$

Note that $g^\alpha(v) \longrightarrow (3, 3, 3)$ as $\alpha \downarrow 0$ and $g^\alpha(v) \longrightarrow (0, 0, 9)$ as $\alpha \to \infty$. This convergence is not monotone as one would possibly expect, since $g^\alpha_2(v)$ attains a maximal value of $g^{\hat{\alpha}}_2(v) \approx 3.0291$ at $\hat{\alpha} = \frac{1}{\ln 2}\,\ln\left[\,\frac{\ln 3 - \ln 2}{\ln 4 - \ln 3}\,\right] \approx 0.4951$.
The results of the analysis of this example are summarised in Figure 1. The yellow simplex represents the space of imputations $\mathbb{I}(v)$, while the Core is the unique imputation depicted as a blue point. The red curve denotes the set of α-Gately values, $\{g^\alpha(v) \mid \alpha > 0\}$.
Finally, we compute the Shapley value as $\phi = \left(3\tfrac{1}{2}\,,\,3\,,\,2\tfrac{1}{2}\right) \notin C(v)$, which cannot be expressed as a proper α-Gately value. ♦

Exploring the relationship of the Gately value with other Core selectors As observed, the Gately value can be defined for any three-player cooperative game and in case of nonempty Core it behaves as a Core selector. There we also discussed the relationship of the Gately value with one of the main Core selectors, the *Core-center* introduced in González-Díaz and Sánchez-Rodríguez (2007). This solution concept, defined only for games with nonempty Core, satisfies properties as efficiency, individual and coalitional rationality, equal treatment of equal, the dummy player property, among others.

The next example addresses the relationship of the Gately value with the Core center concept. The *Core center* of a game v with a non-empty Core $C(v) \neq \emptyset$ is defined as $\mu(v) = \mathbb{E}(C(v))$, being the average of all Core imputations with equal weight or, equivalently, the expected Core allocation. Alternatively, the Core center is the center of gravity of the Core of a game.

Example 4.7 Consider the class of three-player games with $v_i = 0$ for $i = 1, 2, 3$, $v(12) = \alpha$, $v(13) = \beta$, and $v(23) = v(N) = 1$, where $0 \leqslant \alpha \leqslant \beta \leqslant 1$. We compute that $C(v) = \{ (0, t, 1-t) \mid 0 \leqslant \alpha \leqslant t \leqslant 1-\beta \} \neq \emptyset$ for $\alpha + \beta \leqslant 1$.
For $\alpha + \beta \leqslant 1$ we compute the Core center and the Gately value as

$$\mu(v) = \left(0\,, \tfrac{1}{2} - \tfrac{1}{2}(\beta - \alpha)\,, \tfrac{1}{2} - \tfrac{1}{2}(\beta - \alpha) \right)$$

$$g(v) = \left(0\,, \frac{1-\beta}{2-\alpha-\beta}\,, \frac{1-\alpha}{2-\alpha-\beta} \right) \in C(v)$$

For $\alpha + \beta \leqslant 1$, we now identify that $\mu(v) = g(v)$ if and only if $\alpha = \beta$ or $\alpha + \beta = 1$. This shows that coincidence of these two solution concepts is rather accidental. ♦

Another solution concept explicitly designed as Core selector is the *Alexia value* introduced by Tijs et al. (2011). This solution concept only exists for games with nonempty Core, the class of balanced games. Particularly, the Alexia value is obtained by averaging all the so-called *lexinals*, where a lexinal is defined as a lexicographical maximum of the Core with respect to an arbitrary order on the players. On the domain of balanced games, the Alexia value satisfies properties as individual rationality, efficiency, symmetry, and the dummy player property. In the three-player clan game discussed in Example 3.13 of Tijs et al. (2011), we compute that the Gately value is $(4, 4, 2)$, the Nucleolus is $(\frac{15}{4}, \frac{15}{4}, \frac{5}{2})$, and the Alexia value is $(\frac{25}{6}, \frac{25}{6}, \frac{10}{6})$.

We conclude that the Gately value is rather different from many of the Core selectors considered in the literature. Furthermore, the Gately value is defined for the much wider class of regular games than the class of balanced games, which admit non-empty Cores. This justifies a closer look at a comparison with the Shapley value, which is defined over the complete space of all cooperative games. This is explored in the next section.

5 Gately values of n-player games

In this section we look at the relationship of Gately values and the Core as well as the Shapley value of arbitrary n-player games. We first consider the Gately value and its membership of the Core. Subsequently, we consider the relationship between the Shapley and Gately values. The next example illustrates the issues of the questions we investigate here.[12]

Example 5.1 Consider a regular zero-normalised four-player game v with $N = \{1, 2, 3, 4\}$ and $v_i = 0$ for all $i \in N$, $v(12) = 8$, $v(13) = v(14) = v(23) = v(24) = v(34) = 1$, $v(123) = v(124) = 5$, $v(134) = v(234) = 4$, and $v(N) = 12$. We note that the Core of this game is non-empty, since

[12]We remark that Littlechild and Vaidya (1976) already provided an example of a four-player game in which the Gately value is not in the non-empty Core of that game.

$\left(4\frac{1}{2}, 4\frac{1}{2}, 1\frac{1}{2}, 1\frac{1}{2}\right) \in C(v)$.
From these worths, we derive that $M(v) = (8, 8, 7, 7)$. From this it is easy to establish that the Gately value of this game is given by $g = \left(3\frac{1}{5}, 3\frac{1}{5}, 2\frac{4}{5}, 2\frac{4}{5}\right)$. Clearly, $g \notin C(v)$. ♦

In the following analysis we particularly focus on the conditions on n-player games under which $g(v) \in C(v)$. We establish some full characterisations of these equivalences.

5.1 Gately values and the Core

The main condition for which a "symmetric" or "anonymous" cooperative game has a non-empty Core has been identified as the condition that for all coalitions $S \in 2^N : \frac{v(S)}{|S|} \leqslant \frac{v(N)}{n}$ (Shubik, 1982, page 149). This condition has been referred to as "domination by the grand coalition" by Chatterjee et al. (1993) and as "top convexity" by Jackson and van den Nouweland (2005). We generalise this condition to identify when the α-Gately value is in the Core of a regular, zero-normalised cooperative game.

Definition 5.2 *Let $v \in \mathbb{V}^N$ be a semi-standard cooperative game and let $\alpha > 0$. The cooperative game v is said to be α-**top dominant** if for every coalition $S \in 2^N$*

$$\left[v(S) - \sum_{j \in S} v_j \right] \cdot \sum_{j \in N} \left(M_j(v) - v_j \right)^{\alpha} \leqslant \left[v(N) - \sum_{j \in N} v_j \right] \cdot \sum_{j \in S} \left(M_j(v) - v_j \right)^{\alpha} . \tag{22}$$

For $\alpha = 1$ we refer to the started property as "top dominance".

First we remark that $\sum_{j \in S} \left(M_j(v) - v_j \right)^{\alpha} \geqslant 0$ for every semi-standard cooperative game $v \in \mathbb{V}^N$ and every $\alpha > 0$.

Furthermore, the concept of α-top dominance is akin to the notions listed above in the sense that for a semi-standard zero-normalised game $v \in \mathbb{V}^N_0$, property (22) can be rewritten as

$$\frac{v(S)}{\sum_{j \in S} M_j(v)^{\alpha}} \leqslant \frac{v(N)}{\sum_{j \in N} M_j(v)^{\alpha}}$$

for $\sum_{j \in N} M_j(v)^{\alpha} \geqslant \sum_{j \in S} M_j(v)^{\alpha} > 0$. Moreover, implementing $\alpha = 0$, the notion of α-top dominance clearly generalises the notion of top convexity, as top convexity is equivalent to 0-top dominance for zero-normalised games. Indeed, for zero-normalised game $v \in \mathbb{V}^N_0$ we straightforwardly derive $\sum_{j \in S} M^0_j(v) = |S|$, immediately leading to the conclusion that 0-top dominance is the same as top convexity.

The next theorem generalises the insights of Theorem 4.2 to games with arbitrary player sets.

Theorem 5.3 *Let $\alpha > 0$. A standard cooperative game $v \in \mathbb{V}^N$ is α-top dominant if and only if $g^{\alpha}(v) \in C(v)$.*

Proof. Let $v \in \mathbb{V}^N$ be standard and let $\alpha > 0$.
Now $g^{\alpha}(v) \in C(v)$ if and only if it holds that for every coalition $S \in 2^N$: $\sum_{j \in S} g^{\alpha}_j(v) \geqslant v(S)$. This is

equivalent to the condition that for every coalition $S \in 2^N$:

$$\sum_{i \in S} v_i + \frac{\sum_{i \in S} (M_i(v) - v_i)^\alpha}{\sum_{j \in N} \left(M_j(v) - v_j\right)^\alpha} \cdot \left(v(N) - \sum_{j \in N} v_j\right) \geqslant v(S)$$

From v being standard, it follows that $\sum_{i \in N} (M_i(v) - v_i)^\alpha > 0$. Hence, the above is equivalent to the condition that for every coalition $S \in 2^N$:

$$\sum_{i \in S} (M_i(v) - v_i)^\alpha \cdot \left(v(N) - \sum_{j \in N} v_j\right) \geqslant \sum_{j \in N} \left(M_j(v) - v_j\right)^\alpha \cdot \left(v(S) - \sum_{i \in S} v_i\right)$$

This is exactly the α-top dominance property. ■

Properties of top dominant games The next definition introduces a reduced notion of superadditivity that fits with top dominance. This form of superadditivity is defined as "partitional" superadditivity.

Definition 5.4 *A cooperative game $v \in \mathbb{V}^N$ is **partitionally superadditive** if for every coalition $S \subseteq N$ it holds that $v(S) + v(N \setminus S) \leqslant v(N)$.*

The next theorem shows that top dominant games always satisfy regularity as well as the partitional superadditivity property defined above.

Theorem 5.5 *Let $v \in \mathbb{V}^N$ be a standard cooperative game. If the game v is α-top dominant for some $\alpha > 0$, then v is regular as well as partitionally superadditive.*

Proof. Let $v \in \mathbb{V}^N$ be a standard game and let $\alpha > 0$ be such that v is α-top dominant. Hence, $\sum_{j \in N} \left(M_v(j) - v_j\right)^\alpha > 0$ and $\sum_{j \in S} \left(M_v(j) - v_j\right)^\alpha \geqslant 0$ for every coalition $S \subset N$.

We first show that v is essential, together with the hypothesis that v is standard, implying that v is regular.
When we apply the α-top dominance property to the coalition $N - i$ for any $i \in N$ we arrive at

$$\left(v(N-i) - \sum_{j \neq i} v_j\right) \cdot \sum_{j \in N} \left(M_v(j) - v_j\right)^\alpha \leqslant \left(v(N) - \sum_{j \in N} v_j\right) \cdot \sum_{j \neq i} \left(M_v(j) - v_j\right)^\alpha$$

Adding these inequalities over all $i \in N$ we arrive at the conclusion that

$$\begin{aligned}\sum_{i \in N} \left(v(N-i) - \sum_{j \neq i} v_j\right) \cdot \sum_{j \in N} \left(M_v(j) - v_j\right)^\alpha &\leqslant \left(v(N) - \sum_{j \in N} v_j\right) \sum_{i \in N} \sum_{j \neq i} \left(M_v(j) - v_j\right)^\alpha \\ &= (n-1) \left(v(N) - \sum_{j \in N} v_j\right) \sum_{j \in N} \left(M_v(j) - v_j\right)^\alpha\end{aligned}$$

Hence,

$$\sum_{i \in N} \left(v(N-i) - \sum_{j \neq i} v_j\right) \leqslant (n-1) \left(v(N) - \sum_{j \in N} v_j\right)$$

This implies that

$$\sum_{i \in N} M_i(v) \geqslant v(N). \tag{23}$$

Next, suppose to the contrary that $v(N) < \sum_{j \in N} v_j$. From v being a standard game, there is some $i \in N$ with $M_i(v) > v_i$. We can apply the α-top dominance property to $S = \{i\}$ and derive that

$$0 = (v_i - v_i) \cdot \sum_{j \in N} (M_j(v) - v_j)^\alpha \leqslant \left(v(N) - \sum_{j \in N} v_j \right) \cdot (M_i(v) - v_i) < 0$$

which is impossible. Therefore, we conclude that $v(N) \geqslant \sum_{j \in N} v_j$ and, together with (23), we have shown the assertion that v is essential.

Next we show that v is partitionally superadditive.
Let $S \subset N$ be some coalition. Then, from α-top dominance, it holds for S that

$$\left(v(S) - \sum_{j \in S} v_j \right) \cdot \sum_{i \in N} (M_i(v) - v_i)^\alpha \leqslant \left(v(N) - \sum_{j \in N} v_j \right) \cdot \sum_{i \in S} (M_i(v) - v_i)^\alpha$$

$$\left(v(N \setminus S) - \sum_{j \in N \setminus S} v_j \right) \cdot \sum_{i \in N} (M_i(v) - v_i)^\alpha \leqslant \left(v(N) - \sum_{j \in N} v_j \right) \cdot \sum_{i \in N \setminus S} (M_i(v) - v_i)^\alpha$$

Adding these two inequalities leads to the conclusion that

$$\left(v(S) + v(N \setminus S) - \sum_{j \in N} v_j \right) \cdot \sum_{i \in N} (M_i(v) - v_i)^\alpha \leqslant \left(v(N) - \sum_{j \in N} v_j \right) \cdot \sum_{i \in N} (M_i(v) - v_i)^\alpha$$

Since $\sum_{i \in N} (M_i(v) - v_i)^\alpha > 0$ for any $\alpha > 0$, we have shown that

$$v(S) + v(N \setminus S) - \sum_{j \in N} v_j \leqslant v(N) - \sum_{j \in N} v_j$$

and, hence, $v(S) + v(N \setminus S) \leqslant v(N)$. We conclude that v is indeed partitionally superadditive. ■

Theorems 5.5 and 5.3 now immediately imply the following corollary.

Corollary 5.6 *Let $v \in \mathbb{V}^N$ be a standard cooperative game and let $\alpha > 0$. If $g^\alpha(v) \in C(v)$, then v is regular and partitionally superadditive.*

One can ask oneself whether the condition of top dominance can be simplified or linked to other regularity properties of cooperative games. As shown in Theorem 5.5 it is clear that top dominance is closely related to the superadditivity property that is widely used in cooperative game theory. The next example shows that top dominance is actually strictly weaker than superadditivity.

Example 5.7 Consider a regular and zero-normalised three-player game with $N = \{1, 2, 3\}$ and v given by $v_i = 0$ for $i = 1, 2, 3$, $v(12) = v(13) = -1$, $v(23) = 0$ and $v(N) = 1$. We note that v is not

superadditive, since $v_1 + v_2 = 0 > v(12) = -1$.
However, for any $\alpha > 0$ we remark that

$$\frac{v(N)}{M_1(v)^\alpha + M_2(v)^\alpha + M_3(v)^\alpha} = \frac{1}{1+2^{\alpha+1}} > 0$$
$$\frac{v(12)}{M_1(v)^\alpha + M_2(v)^\alpha} = \frac{-1}{1+2^{\alpha}} < 0$$
$$\frac{v(13)}{M_1(v)^\alpha + M_3(v)^\alpha} = \frac{-1}{1+2^{\alpha}} < 0$$
$$\frac{v(23)}{M_2(v)^\alpha + M_3(v)^\alpha} = 0$$

Hence, we conclude that v indeed satisfies α-top dominance for every $\alpha > 0$.
Furthermore, we determine easily that $M_1(v) = 1$ and $M_2(v) = M_3(v) = 2$, leading to the conclusion that for every $\alpha > 0$ the Gately values are given as $g_1^\alpha = \frac{1}{1+2^{\alpha+1}}$ and $g_2^\alpha = g_3^\alpha = \frac{2^\alpha}{1+2^{\alpha+1}}$.
It can also easily be checked that for every $\alpha > 0\colon g^\alpha(v) \in C(v)$. ♦

5.2 Comparing the Gately and Shapley values

The Shapley value (Shapley, 1953) has achieved the status as being the prime solution concept for TU-games. It sets a benchmark for assessing the suitability of alternative solution concepts. In this context it is suitable to consider for what classes of TU-games such alternative solution concepts lead to exactly the same allocation as the Shapley value.

We show that the Shapley and Gately values coincide on narrow, but highly relevant, classes of games that satisfy rather strong regularity properties. Theorem 5.9 below identifies a class of structured games that have rather wide applicability. This class is founded on strong conditions on the class of constituting coalitions and the corresponding unanimity games.

We recall for the benefit of the next analysis that all cooperative games can be represented through the unanimity basis of $\mathbb{V}^N$ based on (6). Hence, every game $v \in \mathbb{V}^N$ can be written as $v = \sum_{S \in \Pi_v} \Delta_S(v) \cdot u_S$, where $\Pi_v = \{S \in 2^N \mid \Delta_S(v) \neq 0\}$ is the class of relevant constituting coalitions and $\Delta_v(S) \neq 0$ is the corresponding Harsanyi dividend of coalition $S \in \Pi_v$ in game v.

We also recall that for any game $v \in \mathbb{V}^N$ with corresponding representation $v = \sum_{S \in \Pi_v} \Delta_S(v) \cdot u_S$, the *Shapley value* of v is defined as $\phi(v) \in \mathbb{R}^N$ given by

$$\phi_i(v) = \sum_{S \in \Pi_v \colon i \in S} \frac{\Delta_v(S)}{|S|} \qquad \text{for every } i \in N. \tag{24}$$

The next definition introduces some relevant classes of regular games.

Definition 5.8 *Let $N = \{1, \ldots, n\}$ be a set of players and let $k \in \{2, \ldots, n-1\}$. A game $v \in \mathbb{V}^N$ on N is denoted as a* **k-*game*** *if v is regular and v can be written as $v = \sum_{S \in \Pi_v} \Delta_v(S)\, u_S$ such that $|S| = k$ for all constituting coalitions $S \in \Pi_v$. The subclass of k-games on N is denoted by $\mathbb{V}_k^N \subset \mathbb{V}_\star^N$.*

The subclass of 2-games has been investigated in the literature on its properties. In particular, van den Nouweland et al. (1996) and van den Brink et al. (2023) show that for 2-games the Shapley value coincides with the Nucleolus and the τ-value. It might not be a surprise that we can show for all subclasses of k-games the property that the Gately value coincides with the Shapley value.

Theorem 5.9 *Let $N = \{1, \dots, n\}$ be a set of players and let $k \in \{2, \dots, n-1\}$. For every k-game $v \in \mathbb{V}_k^N$ it holds that $g(v) = \phi(v)$.*

Proof. Let $N = \{1, \dots, n\}$ be a set of players and let $k \in \{2, \dots, n-1\}$. First, we remark that for $k \geqslant 2$ all k-games are zero-normalised by definition.
Now, let $v \in \mathbb{V}_k^N$ be a k-game. Hence, $|S| = k$ for all $S \in \Pi_v$. Assume that $|\Pi_v| = m$ is the number of the constituting coalitions of the game v.
Next, we introduce some additional notation. For every $i \in N$ we let $\Pi_i = \{S \in \Pi_v \mid i \in S\}$. Then the Shapley value of player $i \in N$ can be written as

$$\phi_i(v) = \sum_{S \in \Pi_i} \frac{\Delta_v(S)}{|S|} = \tfrac{1}{k} \sum_{S \in \Pi_i} \Delta_v(S) = \frac{\Delta_i}{k} \qquad \text{where } \Delta_i = \sum_{S \in \Pi_i} \Delta_v(S).$$

Regarding the determination of the Gately value, we note that the marginal contribution of $i \in N$ is now given by

$$M_i(v) = v(N) - v(N-i) = \sum_{S \in \Pi_v} \Delta_v(S) - \sum_{S \in \Pi_v :\, i \notin S} \Delta_v(S) = \sum_{S \in \Pi_i} \Delta_v(S) = \Delta_i.$$

Furthermore, this implies that $\sum_{j \in N} M_i(v) = \sum_{j \in N} \Delta_j = k \cdot v(N)$. Hence, we have determined that for $i \in N$:

$$g_i(v) = \frac{M_i(v)}{\sum_{j \in N} M_j(v)} \cdot v(N) = \frac{\Delta_i}{k \cdot v(N)} \cdot v(N) = \frac{\Delta_i}{k} = \phi_i(v).$$

This shows the assertion of the theorem. ■

Based on Theorem 1 and Corollary 1 of van den Brink et al. (2023) in combination with Theorem 5.9, we can determine another characterisation of the Gately value for the subclass of 2-games:

Corollary 5.10 *On the subclass of 2-games $\mathbb{V}_2^N$, the Gately value is the unique value that satisfies the balanced externalities property in the sense that for every $v \in \mathbb{V}_2^N$ with $v = \sum_{S \in \Pi_v} \Delta_v(S)\, u_S$:*

$$g_i(v) = \sum_{j \neq i} \big(g_j(v) - g_j(v^{-i}) \big) \tag{25}$$

where $v^{-i} = \sum_{T \in \Pi_v^{-i}} \Delta_v(T)\, u_T$ where $\Pi_v^{-i} = \{T \in \Pi_v \mid i \notin T\}$.

The reverse of Theorem 5.9 does not hold. There are other classes of games with strong regularity properties that are not k-games and on which the Gately value coincides with the Shapley value. The next proposition introduces such a subclass of highly regular games.

Proposition 5.11 *Let $v \in \mathbb{V}^N$ be a regular game written as $v = \sum_{S \in \Pi_v} \Delta_v(S)\, u_S$. Assume that $n = k \cdot m$, where $k, m \in \mathbb{N}$ with $k \neq m$, such that $\Pi_v = \Pi^k \cup \Pi^m$, where Π^k is a partitioning of N into m sets of size k and Π^m is a partitioning of N into k sets of size m.*
If there exists some $\Delta \neq 0$ such that $\Delta_v(S) = \Delta$ for all $S \in \Pi_v$, then $g(v) = \phi(v)$.

Proof. Let $i \in N$ be an arbitrary player.
Due to the structure of the game, player $i \in N$ is member of exactly one coalition $S \in \Pi^k$ and one coalition $T \in \Pi^m$. Hence, it easily follows that $\phi_i(v) = \frac{\Delta}{k} + \frac{\Delta}{m}$.
To compute the Gately value, we note that $M_i(v) = 2\Delta$ and $\sum_{j \in N} M_j(v) = n \cdot 2\Delta = 2km\Delta$. Furthermore, $v(N) = (k+m)\Delta$, so the Gately value for player $i \in N$ can be computed as

$$g_i(v) = \frac{2\Delta}{2km\Delta}\,(k+m)\Delta = \frac{k+m}{km}\Delta = \frac{\Delta}{k} + \frac{\Delta}{m} = \Phi_i(v).$$

This shows the assertion. ■

For games that do not satisfy the property stated in Theorem 5.9, the Gately value is only very rarely equal to the Shapley value. The next example discusses a convex five-player game in which the Gately value is not in the Core, while the Shapley value is a Core selector (Shapley, 1971).

Example 5.12 Let $N = \{1,2,3,4,5\}$ and let $v = u_{12} + 3u_{345}$. This game is convex with $v(N) = 4$ and the marginal contribution vector $M = (1,1,3,3,3)$. It is easy to see that the Shapley value is given by $\phi = \left(\frac{1}{2}, \frac{1}{2}, 1, 1, 1\right)$ and the Gately value is given by $g = \left(\frac{4}{11}, \frac{4}{11}, \frac{12}{11}, \frac{12}{11}, \frac{12}{11}\right)$.
We note that $\phi \in C(v)$, while $g \notin C(v)$ since $g_1 + g_2 = \frac{8}{11} < 1 = v(12)$.
We remark that the game v discussed here does not satisfy either of the descriptions introduced in Theorem 5.9 and Proposition 5.11, but that the game has a structure that is similar to the structure of the class of games considered in Proposition 5.11. ♦

6 Concluding remarks

The main contribution of this paper is the investigation of the relationship between the Gately value and other solution concepts on the class of regular cooperative games. In particular, we explore Gately's original impetus for defining the Gately value, namely as a Core selector ,and under what conditions the Gately value, certain Core selectors, and the Shapley value coincide.

We have also proposed and investigated a generalisation of Gately's conception in which the definition of a player's propensity to disrupt is modified by imposing a weight on the denominator. This weight represents an *intensity parameter* that measures how much more or less weight a player puts on her own loss in relation to the loss of all other players.

The provided axiomatisation (Theorem 3.12) is clearly founded on the axiomatisation of the τ-value by Tijs (1987). The τ-value was further axiomatised by Calvo et al. (1995). The latter contribution shows that in general axiomatisations of compromise values—such as the τ-value as well as the Gately value considered here—can be rather intractable and unsatisfactory. It is an open research question whether an accessible axiomatisation of the Gately value and other compromise values can be devised that go beyond the straightforward axiomatisation framework introduced by Tijs (1987).

In Gilles and Mallozzi (2023), we apply the Gately value to develop and study an innovative method to measure centrality in social networks consisting of directed relationships. In this methodology, a social network is converted into a cooperative game by assigning the number of external relationships of each coalition as a worth to that coalition. These are known as *successor representations* of these social networks.

Subsequently, a cooperative game theoretic solution concept can be applied to the successor representation to define a centrality measure. The Shapley value results in the β-centrality measure (van den Brink and Gilles, 2000), while in Gilles and Mallozzi (2023) we apply the Gately value to the successor representation to arrive at the Gately centrality measure. Subsequent analysis shows that the Gately centrality measure can be fully axiomatised on the subclass of hierarchical social networks. It is shown that the Gately measure deviates in one important characteristic from the β-measure, the restricted proportionality property. On the other hand, the Gately measure coincides with the β-measure on the subclass of weakly regular hierarchical networks (Theorem 5).

This application of the Gately value shows that Gately's approach leads to innovative and fruitful ways to investigate social situations beyond the confines of TU-games only.

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