Game theoretic foundations of the Gately power measure for directed networks*

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Abstract

We introduce a new network centrality measure founded on the Gately value for cooperative games with transferable utilities. A directed network is interpreted as representing control or authority relations between players—constituting a *hierarchical* network. The power distribution of a hierarchical network can be represented through a TU-game. We investigate the properties of this TU-representation and investigate the Gately value of the TU-representation resulting in the Gately power measure. We establish when the Gately measure is a Core power gauge, investigate the relationship of the Gately with the β-measure, and construct an axiomatisation of the Gately measure.

Keywords: Cooperative game; authority network; network centrality measure; Gately measure; axiomatisation.

JEL classification: A14; C71; D20

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1 Introduction

The concept of *network centrality* has emerged from sociology, social network analysis and network science (Newman, 2010; Barabási, 2016) into the field of cooperative game theory, giving rise to game theoretic methods to measure the most important and dominant nodes in a hierarchical social network (Redhead and Power, 2022). The underlying method is to construct a cooperative game theoretic representation of characterising features of a network and to apply cooperative game theoretic analysis to create centrality measures for these networks.[1]

We limit ourselves to directed networks as representations of collections of hierarchical or control relationships between the constituting players in a network. Such hierarchical relationships can be found in employment dynamics between managers and subordinates, the interaction between a professor and her students, rivalry between different sports teams based on past performance between them, or the connections between a government agent and the individuals they oversee. We refer to these relationships as "hierarchical" since it implies that the predecessor node has some level of control or authority over the successor node.[2] Taking this interpretation central, we refer to these directed networks as *hierarchical*.

A hierarchical relationship is between a *predecessor* and a *successor*, where the predecessor exercises some form of control or authority over the successor. The most natural representation is that through a TU-game that assigns to every group of players their "total number of successors", which can be interpreted in various ways. We consider the two standard ways: Simply counting the successors of all members of the group, i.e., the number of players that have at least one predecessor that is member of the group; or, counting the number of players for which *all* predecessors are member of that group.[3] We show that these two TU-representations are dual games. We remark that van den Brink and Borm (2002) already characterised the main "strong" successor representation as a convex game (Shapley, 1971), implying that its dual "weak" successor representation is a concave game.

A *power gauge* for a network is now introduced as a vector of weights that are assigned to players in the hierarchical network that represent or measure each player's authority in that network. A *power measure* is now introduced as a map that assigns to every hierarchical network a single power gauge. In this paper we investigate some power measures that assign such gauges founded on game theoretic principles related to the two TU-representations of hierarchical networks considered here. In particular, each simple hierarchical network—in which each player has at most one predecessor—has a natural power gauge in the form of the outdegree of each node in the network, representing the number of successors of a player in that network.

Application of the Shapley value (Shapley, 1953) to the successor representations results in the

[1] We refer to Tarkowski et al. (2018) for an overview of the literature on cooperative game theoretic constructions of centrality measures in directed networks.

[2] This authority can also be psychological and be influential on reputational features in the relationship. An example might be the relationship between two chess players. One of these players can have a psychological advantage over the other based on outcomes of past games between the two players and/or the Elo rating differential between the two players.

[3] As Tarkowski et al. (2018) point out, the successor representations are only one type of representation of a hierarchical network. More advanced TU-representations have been pursued by Gavilán et al. (2023).

β-power measure (van den Brink and Gilles, 1994). This is the centre of the set of Core power gauges for each network. In the β-measure the weight of a player is equally divided among its predecessors. As such it has a purely individualistic foundation to measuring power.

Applying the Gately value of the successor representations results in a fundamentally different conception of a power measure. Here, the set of dominated nodes is treated as a *collective resource* that is distributed according to a chosen principle. In the Gately measure this is the proportional distribution.[4] This stands in contrast to the individualistic perspective of the β-measure.

Since, the Gately measure is founded on such different principles, it is not a surprise that the assigned Gately power gauges are not necessarily Core power gauges. We identify conditions under which the assigned Gately power gauge is a Core power gauge in Theorem 3.3. In particular, we show that for the class of (weakly) regular hierarchical networks, the Gately power measure assigns a Core power gauge for that network.

We are able to devise an axiomatic characterisation of the Gately value as the unique power measure that satisfies three properties. First, it is normalised to the number of nodes that have predecessors, which is satisfied by many other power measures as well. Second, it satisfies "normality" which imposes that a power measure assigns the full weight of controlling successors with no other predecessors and the power measure of the reduced network with only those nodes that have multiple predecessors. Finally, it satisfies a proportionality property in the sense that the power measure assigned is proportional to how many successors a node has.

Finally, we address the question under which conditions the β- and Gately power measures are equivalent. We show that for the class of weakly regular hierarchical networks this equivalence holds. This is exactly the class of networks for which the Gately measure assigns a Core power gauge. This insight cannot be reversed, since there are non-regular networks for which the Gately and β-measures are equivalent.

Relationship to the literature The study of centrality in networks has evolved to be a significant part of network science (Newman, 2010; Barabási, 2016). In economics and the social sciences there has been a focus on Bonacich centrality in social networks. This centrality measure is founded on the eigenvector of the adjacency matrix that represents the network (Bonacich, 1987). In economics this has been linked to performance indicators of network representations of economic interactions such as production networks (Ballester et al., 2006; Huremovic and Vega-Redondo, 2016; Allouch et al., 2021). The nature of these networks is that they are undirected and, therefore, fundamentally different from the hierarchical networks considered here.

Traditionally, the investigation of directed networks focussed on *degree centrality*—measuring direct dominance relationships (van den Brink and Gilles, 2003; van den Brink and Rusinowska, 2022)—and on *betweenness centrality*, which considers the position of nodes in relation to membership of (critical) pathways in the directed network (Bavelas, 1948; White and Borgatti, 1994; Newman, 2005; Arrigo et al., 2018).

Authority and control in networks has only more recently been investigated from different

[4] We remark that other distribution principles can also be applied such as distributions founded on egalitarian fairness considerations. This falls outside the scope of the present paper.

perspectives. Yang-Yu et al. (2012) considers an innovative perspective founded on control theory. More prevalent is the study of centrality in hierarchical networks through the β-measure and its close relatives. van den Brink and Gilles (1994) introduced the β-measure as a natural measure of influence and considered some non-game theoretic characterisations. The β-measure is closely related to the PageRank measure introduced by Brin and Page (1998) and considered throughout the literature on social network centrality measurement.

The β-measure has been linked to game theoretic measurement of centrality in directed networks by van den Brink and Gilles (2000) and van den Brink and Borm (2002). The β-measure was identified as the Shapley value of the standard successor representations as TU-representations of domination and control in directed networks. van den Brink et al. (2008) develop this further through additional characterisations. Gavilán et al. (2023) introduce other, more advanced TU-representations of directed networks and study their Shapley values. They consider a family of centrality measures resulting from this methodology.

Gómez et al. (2003), del Pozo et al. (2011) and Skibski et al. (2018) introduce and explore a game theoretic methodology for measuring network power that is fundamentally different from the methodology used in this paper and the literature reviewed above. These authors consider a well-chosen TU-game on a networked population of players and subsequently compare the allocated payoffs based on the Shapley value in the unrestricted game with the Shapley value of the network-restricted TU-game. The normalisation of the generated differences now exactly measure the network-positional effects on the players, which can be interpreted as a centrality measure.

Finally, with regard to the Gately value as a solution concept for TU-games, this conception was seminally introduced for some specific 3-player cost games by Gately (1974). This contribution inspired a further development of the underlying conception of "propensity to disrupt" by Littlechild and Vaidya (1976) and Charnes et al. (1978), including the definition of several related solution concepts. Littlechild and Vaidya (1976) also developed an example of a 4-player TU-game in which the Gately value is not a Core imputation. More recently, Staudacher and Anwander (2019) generalised the scope of the Gately value and identified exact conditions under which this value is well-defined. This has further been developed by Gilles and Mallozzi (2023), which showed that the Gately value is always a Core imputation for 3-player games, devised an axiomatisation for the Gately value for arbitrary TU-games, and introduced a generalised Gately value founded on weighted propensities to disrupt.

Structure of the paper Section 2 discusses the foundations of the game theoretic approach that is pursued in this paper. It defines the successor representations and presents their main properties. Furthermore, the standard solution concepts of the Core and the Shapley value are applied to these successor representations. Section 3 introduces the Gately measure, which represents a different philosophy of measuring the exercise of control and power in a network. We investigate when the Gately measure assigns a Core power gauge to a network and we devise an axiomatisation of the Gately measure. The paper concludes with a comprehensive comparison of the Gately and β-measures, identifying exact conditions under which these two measures are equivalent.

2 Game theoretic representations of hierarchical networks

In our study, we focus on networks with directed links, where each link has specifically the interpretation of being the representation of a hierarchical relationship. In a directed network, the direction of a link indicates that one node is positioned as a predecessor while the other node is considered a successor in that particular relationship. Here we interpret this explicitly as a control or authority relationship. Therefore, we denote these networks as *hierarchical* throughout this paper.

In hierarchical networks, predecessors exercise some form of authority over its successors, allowing the assignment of that control to that particular node. This results in a natural game theoretic representation. We explore these game theoretic representations in this section and investigate the properties of these games.

Notation: Representing hierarchical networks Let $N = \{1, \dots, n\}$ be a finite set of nodes, where $n \in \mathbb{N}$ is the number of nodes considered. Usually we assume that $n \geqslant 3$. A *hierarchical network* on N is a map $D \colon N \to 2^N$ that assigns to every node $i \in N$ a set of *successors* $D(i) \subseteq N \setminus \{i\}$. We explicitly exclude that a node succeeds itself, i.e., $i \notin D(i)$. The class of all directed networks on node set N is denoted as $\mathbb{D}^N = \{D \mid D \colon N \to 2^N \text{ with } i \notin D(i) \text{ for all } i \in N\}$.[5]

Inversely, in a directed network $D \in \mathbb{D}^N$, for every node $i \in N$, the subset $D^{-1}(i) = \{j \in N \mid i \in D(j)\}$ denotes the set of its *predecessors* in D. Due to the general nature of the networks considered here, we remark that it might be the case that $D(i) \cap D^{-1}(i) \neq \varnothing$, i.e., some nodes can be successors as well as predecessors of a node.

We introduce the following additional notation to count the number of successors and predecessors of a node in a network $D \in \mathbb{D}^N$:

(i) The map $s_D \colon N \to \mathbb{N}$ counts the number of successors of a node defined by $s_D(i) = \# D(i)$ for $i \in N$;

(ii) The map $p_D \colon N \to \mathbb{N}$ counts the number of predecessors of a node defined by $p_D(i) = \# D^{-1}(i)$ for $i \in N$;

The previous analysis leads to a natural partitioning of the node set N into different classes based on the number of predecessors of the nodes in a given network $D \in \mathbb{D}^N$:

$$N_D^o = \{i \in N \mid D^{-1}(i) = \varnothing\} = \{i \in N \mid p_D(i) = 0\}$$
$$N_D = N \setminus N_D^o = \{i \in N \mid p_D(i) \geqslant 1\}$$
$$N_D^a = \{i \in N \mid p_D(i) = 1\}$$
$$N_D^b = \{i \in N \mid p_D(i) \geqslant 2\}$$

Note that $N = N_D^o \cup N_D$, $N_D = N_D^a \cup N_D^b$. In particular, $\{N_D^o, N_D^a, N_D^b\}$ forms a partitioning of the node set N. We introduce counters $n_D = \# N_D$, $n_D^a = \# N_D^a$ and $n_D^b = \# N_D^b$, leading to the conclusion

[5] We emphasise that in our setting, hierarchical networks are not necessarily tiered or top-down. Hence, we allow these networks to contain cycles and even binary relationships. This allows the incorporation of sports competitions and other social activities to be represented by these hierarchical networks.

that $n_D = n_D^a + n_D^b$.

The constructed partitioning informs the analysis of the game theoretic representation of the hierarchical authority structure imposed by D on the node set N. Our analysis will show that for certain centrality measures, the class of nodes that have multiple predecessors N_D^b plays a critical role in the determination of the assignment of a power index to these predecessors.

The partitioning of the node set N based on the structure imposed by $D \in \mathbb{D}^N$ allows further notation to be introduced for every node $i \in N$:

- $s_D^a(i) = \#\left[D(i) \cap N_D^a \right]$ and $s_D^b(i) = \#\left[D(i) \cap N_D^b \right]$, resulting in the conclusion that $s_D(i) = s_D^a(i) + s_D^b(i)$.
- From the definitions above we conclude immediately that

$$\sum_{i \in N} s_D^a(i) = \# N_D^a$$

and

$$\sum_{i \in N} s_D^b(i) = \sum_{j \in N_D^b} p_D(j).$$

Classes of hierarchical networks The next definition introduces some normality properties on hierarchical networks that will be used for certain theorems.

Definition 2.1 *Let $D \in \mathbb{D}^N$ be some hierarchical network on node set N.*

(a) *The network D is **weakly regular** if for all nodes $i, j \in N_D^b$: $p_D(i) = p_D(j)$.*
The collection of weakly regular hierarchical networks is denoted by $\mathbb{D}_w^N \subset \mathbb{D}^N$.

(b) *The network D is **regular** if for all nodes $i, j \in N_D$: $p_D(i) = p_D(j)$.*
The collection of regular hierarchical networks is denoted by $\mathbb{D}_r^N \subset \mathbb{D}_w^N$.

(c) *The network D is **simple** if for every node $i \in N_D$: $p_D(i) = 1$.*
The collection of simple hierarchical networks is denoted by $\mathbb{D}_s^N \subset \mathbb{D}_r^N$.

In a regular network, each node has either no predecessors, or a given fixed number of predecessors. Hence, all nodes with predecessors have exactly the same number of predecessors. In a weakly regular network, each node has either no predecessors, or exactly one predecessors, or a given fixed number $p \geqslant 2$ of predecessors.

The notion of a simple network further strengthens the requirement of a regular network. It imposes that all nodes either have no predecessors, or exactly one predecessor.

Furthermore, van den Brink and Borm (2002) introduced the notion of a *simple subnetwork* of a given network $D \in \mathbb{D}^N$ on the node set N. We elaborate here on that definition.

Definition 2.2 *Let $D \in \mathbb{D}^N$ be a given hierarchical network on N.*
*A network $T \in \mathbb{D}^N$ is a **simple subnetwork** of D if it satisfies the following two properties:*

(i) *For every node* $i \in N\colon T(i) \subseteq D(i)$, *and*

(ii) *For every node* $j \in N_D\colon p_T(j) = 1$.

The collection of a simple subnetworks of D *is denoted by* $\mathcal{S}(D)$.

The collection of simple subnetworks of a given network can be used to analyse the Core of the game theoretic representations of hierarchical games as shown below. It is easy to establish that a hierarchical network D is simple if and only if $\mathcal{S}(D) = \{D\}$.

2.1 Game theoretic representations of hierarchical networks

Using the notation introduced above, we are able to device cooperative game theoretic representations of hierarchical networks. We recall that a *cooperative game with transferable utilities*—or a TU-game—on the node set N is a map $v\colon 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$. A TU-game v assigns to every group of nodes $H \subseteq N$ a certain "worth" $v(H) \in \mathbb{R}$. A group of nodes $H \subseteq N$ is also denoted as a *coalition* of nodes, to use a more familiar terminology from cooperative game theory.

To embody the control or authority represented by a hierarchical network $D \in \mathbb{D}^N$ on the node set N as a cooperative game, we introduce some additional notation. For every group of nodes $H \subseteq N$ we denote

$$D(H) = \{j \in N \mid D^{-1}(j) \cap H \neq \varnothing\} = \cup_{i \in H} D(i) \tag{1}$$

as the *(weak) successors* of coalition H in D. A node is a (weak) successor of a node group if at least one of its predecessors is a member of that group.

Similarly, we introduce

$$D^*(H) = \{j \in N \mid \varnothing \neq D^{-1}(j) \subseteq H\} = \{j \in N_D \mid D^{-1}(j) \subseteq H\} \tag{2}$$

as the *strong successors* of coalition H in D. A node is a strong successor of a node group if *all* predecessors of that node are members of that group. Clearly, strong successors of a node group are completely controlled by the nodes in that particular group and full control can be exercised. This compares to regular or weak successors of a node group over which the nodes in that group only exercise partial control.

The next definition introduces the two main cooperative game theoretic embodiments of this control over other nodes in a network.

Definition 2.3 *Let* $D \in \mathbb{D}^N$ *be some hierarchical network on node set* N.

(a) *The* ***successor representation*** *of* D *is the TU-game* $s_D\colon 2^N \to \mathbb{N}$ *for every coalition* $H \subseteq N$ *given by* $s_D(H) = \#D(H)$, *the number of successors of the coalition* H *in the network* D.

(b) *We additionally introduce two* ***partial successor representations*** *as the two TU-games* $s^a_D, s^b_D\colon 2^N \to \mathbb{N}$, *which for every coalition* $H \subseteq N$ *are given by* $s^a_D(H) = \#\left[\, D(H) \cap N^a_D \,\right]$ *and* $s^b_D(H) = \#\left[\, D(H) \cap N^b_D \,\right]$.

(c) *The **strong successor representation** of D is the TU-game $\sigma_D \colon 2^N \to \mathbb{N}$ for every coalition $H \subseteq N$ given by $\sigma_D(H) = \# D^*(H)$, the number of strong successors of the coalition H in the network D.*

The successor representation is also known as the "successor game" in the literature and the strong successor representation σ_D as the "conservative successor game" on D (Gilles, 2010). It is clear that the four TU-games introduced in Definition 2.3 embody different aspects of the control exercised over nodes in a given hierarchical network. In particular, these TU-games count the number of successors that are under the control of nodes in a selected coalition.

Properties of successor representations The next list collects some simple properties of these four games introduced here.

Proposition 2.4 *Let $D \in \mathbb{D}^N$ be some hierarchical network on node set N. Then the following properties hold regarding the successor representations s_D, s^a_D, s^b_D and σ_D:*

(i) *For every node $i \in N \colon s_D(\{i\}) = s_D(i)$ and the worth of the whole node set is determined as $s_D(N) = n_D = \# N_D$.*

(ii) $s_D = s^a_D + s^b_D$.

(iii) *For every coalition $H \subseteq N \colon s^a_D(H) = \sum_{i \in H} s^a_D(i)$, implying that the partial successor representation s^a_D is an additive game.*

(iv) *For every coalition $H \subseteq N \colon s^b_D(H) \leqslant \sum_{i \in H} s^b_D(i)$.*

(v) *$\sigma_D = s^a_D + \hat{\sigma}_D$ where for every coalition $H \subseteq N \colon \hat{\sigma}_D(H) = \sigma_D(H) - s^a_D(H) \leqslant s^b_D(H)$.*

(vi) *For every node $i \in N \colon \sigma_D(\{i\}) = s^a_D(i)$ and the worth of the whole node set is determined as $\sigma_D(N) = n_D = \# N_D$.*

These properties follow straightforwardly from the definitions, therefore a proof is omitted.

The next theorem collects some properties of the successor representations that have not been remarked explicitly in the literature on cooperative game theoretic approaches to representations of hierarchical networks.[6]

Theorem 2.5 *Let $D \in \mathbb{D}^N$ be some hierarchical network on node set N. Then the following properties hold for the successor representations s_D and σ_D:*

(i) *The strong successor representation σ_D is the dual of the successor representation s_D in the sense that*

$$\sigma_D(H) = s_D(N) - s_D(N \setminus H) \qquad \text{for all } H \subseteq N. \tag{3}$$

[6] We recall that the *unanimity game* of coalition $H \neq \varnothing$ is defined by $u_H \colon 2^N \to \{0, 1\}$ such that $u_H(T) = 1$ if and only if $H \subseteq T \subseteq N$. This implies that $u_H(T) = 0$ for all other coalitions $T \subseteq N$.

(ii) *The strong successor representation is decomposable into unanimity games with*

$$\sigma_D = \sum_{j \in N_D} u_{D^{-1}(j)}. \tag{4}$$

(iii) *The strong successor representation σ_D is a convex TU-game (Shapley, 1971) in the sense that $\sigma_D(H) + \sigma_D(K) \leqslant \sigma_D(H \cup K) + \sigma_D(H \cap K)$ for all $H, K \subseteq N$.*

(iv) *The successor representation s_D is concave in the sense that $s_D(H) + s_D(K) \geqslant s_D(H \cup K) + s_D(H \cap K)$ for all $H, K \subseteq N$*

Proof. Let $D \in \mathbb{D}^N$ be some hierarchical network on node set N and let the TU-games s_D and σ_D be as defined in Definition 2.3.
To show assertion (i), let $H \subseteq N$, then it holds that

$$\begin{aligned} s_D(N) - s_D(N \setminus H) &= n_D - \#D(N \setminus H) = n_D - \#\{j \in N \mid D^{-1}(j) \cap (N \setminus H) \neq \varnothing\} \\ &= n_D - \#\{j \in N \mid D^{-1}(j) \setminus H \neq \varnothing\} = \#\{i \in N_D \mid D^{-1}(i) \setminus H = \varnothing\} \\ &= \#\{i \in N_D \mid D^{-1}(i) \subseteq H\} = \sigma_D(H). \end{aligned}$$

This shows that σ_D is indeed the dual game of s_D.

Assertion (ii) is Lemma 2.2 in van den Brink and Borm (2002) and assertion (iii) follows immediately from (ii). Finally, assertion (iv) is implied by the fact that s_D is the dual game of σ_D—following from assertion (i)—and σ_D is convex. ∎

The duality between the successor representation and the strong successor representation implies that some cooperative game theoretic solution concepts result in exactly the same outcomes for both games. In particular, we refer to the Core, the Weber set, the Shapley value, and the Gately value of these successor representations as explored below.

2.2 Some standard solutions of the successor representations

The cooperative game theoretic approach to measuring power or hierarchical centrality is based on the assignment of a quantified control gauge to every individual node in a given hierarchical network. A power or hierarchical centrality measure now refers to a rule or procedure that assigns to every node in any hierarchical network such a gauge. In this section we set out the foundations for this approach.

Definition 2.6 *Let $D \in \mathbb{D}^N$ be some hierarchical network. A **power gauge** for D is a vector $\delta \in \mathbb{R}^N_+$ such that $\sum_{i \in N} \delta_i = n_D$.*
*A **power measure** on $\mathbb{D}^N$ is a function $m \colon \mathbb{D}^N \to \mathbb{R}^N_+$ such that $\sum_{i \in N} m_i(D) = n_D$ for every hierarchical network $D \in \mathbb{D}^N$.*

The normalisation of a power gauge for a network $D \in \mathbb{D}^N$ to the allocation of the total number of nodes in N_D is a yardstick that is adopted in the literature, which we use here as well. This normalisation is in some sense arbitrary, but it allows a straightforward application of the cooperative game theoretic methodology as advocated here.

The game theoretic approach adopted here allows us to apply basic solution concepts to impose well-accepted properties on power gauges and power measures. The well-known notion of the Core of a TU-game imposes lower bounds on the power gauges in a given hierarchical network. This leads to the following notion.

Definition 2.7 *A **Core power gauge** for a given hierarchical network $D \in \mathbb{D}^N$ is a power gauge $\delta \in \mathbb{R}^N_+$ which satisfies that for every group of nodes $H \subseteq N$: $\sum_{j \in H} \delta_j \geqslant \sigma_D(H) = \# D^*(H)$.*
The set of Core power gauges for D is denoted by $C(D) \subset \mathbb{R}^N_+$.

The Core requirements on a power gauge impose that every group of nodes is collectively assigned at least the number of nodes that it fully controls. This seems a rather natural requirement. The following insight investigates the structure of the set of Core power gauges for a hierarchical network.

Proposition 2.8 *Let $D \in \mathbb{D}^N$ be some hierarchical network on node set N. Then the following hold:*

(i) *If D is a simple hierarchical network, then there exists a unique Core power gauge, $C(D) = \{\delta^D\}$, where $\delta^D_i = s_D(i)$ for every node $i \in N$.*

(ii) *More generally, $C(D)$ is equal to the Weber set of σ_D, which is the convex hull of the unique Core power gauges of all simple subnetworks of D given by $C(D) = \mathrm{Conv}\, \{\delta^T \mid T \in \mathcal{S}(D)\} \neq \varnothing$.*

Proof. Let D be a simple hierarchical network. Hence, $p_D(i) = 1$ for all $i \in N_D$. Therefore, for every group of nodes $H \subseteq N$ it holds that $\sigma_D(H) = s_D(H) = \sum_{j \in H} s_D(j)$. Therefore, δ^D as defined above satisfies the Core requirement for every $H \subseteq N$.
Furthermore, suppose that $\delta \in C(D)$. Then from $\delta_i \geqslant s_D(i) = \delta^D_i$ for every node $i \in N$ and $\sum_{j \in N} \delta_j = n_D = \sum_{j \in N} \delta^D_j$ it immediately follows that $\delta_i = \delta^D_i$ for all $i \in N$. This shows that δ^D is the unique Core power gauge for the simple hierarchical network D, showing assertion (i).

Assertion (ii) follows immediately from Theorem 4.2 in van den Brink and Borm (2002) in combination with assertion (i). ■

The β-measure A well-established power measure for hierarchical networks was first introduced by van den Brink and Gilles (1994) and further developed in van den Brink and Gilles (2000) and van den Brink et al. (2008). This *β-measure* is for every node $i \in N$ defined by

$$\beta_i(D) = \sum_{j \in D(i)} \frac{1}{p_D(j)} = s^a_D(i) + \sum_{j \in D(i) \cap N^b_D} \frac{1}{p_D(j)} \tag{5}$$

The following proposition collects the main insights from the literature on the β-measure.

Proposition 2.9 *Let $D \in \mathbb{D}^N$ be a hierarchical network. Then the following properties hold:*

(i) *$\beta(D) \in C(D)$ is the geometric centre of the set of Core power gauges of D.*

(ii) $\beta(D) = \varphi(s_D) = \varphi(\sigma_D)$, *where φ is the Shapley value*[7] *on the collection of all cooperative games on N.*

3 The Gately power measure

Let $v\colon 2^N \to \mathbb{R}$ be a TU-game on the node set N with $\sum_{i \in N} v(\{i\}) \leqslant v(N) \leqslant \sum_{i \in N} M_i(v)$ where $M_i(v) = v(N) - v(N - i)$. Then the *Gately value* of the game v is given by $g(v) \in \mathbb{R}^N$, which is defined for every node $i \in N$ by

$$g_i(v) = v(\{i\}) + \frac{M_i(v) - v(\{i\})}{\sum_{j \in N} \left(M_j(v) - v(\{j\}) \right)} \left[v(N) - \sum_{j \in N} v(\{j\}) \right] \tag{6}$$

The Gately value was seminally introduced by Gately (1974) and further developed by Littlechild and Vaidya (1976), Charnes et al. (1978), Staudacher and Anwander (2019) and Gilles and Mallozzi (2023).

We apply the Gately value to the two successor representations formulated above. We show that, similar to the β-measure, both the regular successor representation and the conservative successor representation result in the same Gately value, defining the *Gately power measure.*

Theorem 3.1 *Let $D \in \mathbb{D}^N$ be a directed network on node set N. Then*

$$g\left(s_D\right) = g\left(\sigma_D\right) = \xi(D) \tag{7}$$

where $\xi\colon \mathbb{D}^N \to \mathbb{R}^N$ is introduced as the ***Gately power measure*** *on the class of hierarchical networks $\mathbb{D}^N$ on N with*

$$\xi_i(D) = \begin{cases} s^a_D(i) + \frac{s^b_D(i)}{\sum_{j \in N^b_D} p_D(j)}\, n^b_D & \text{if } N^b_D \neq \varnothing \\ s^a_D(i) & \text{if } N^b_D = \varnothing \end{cases} \tag{8}$$

for every node $i \in N$.
Furthermore, the Gately measure ξ is the unique power measure that balances the propensities to disrupt a network given by

$$\frac{s^b_D(i)}{s_D(i) - \xi_i(D)} = \frac{s^b_D(j)}{s_D(j) - \xi_j(D)} \tag{9}$$

over all nodes $i, j \in N_D$.

Proof. Let $D \in \mathbb{D}^N$ be such that $N^b_D \neq \varnothing$. Then its successor representation s_D is characterised for

[7] It is well-established that every TU-game $v\colon 2^N \to \mathbb{R}^N$ can be written as $v = \sum_{H \subseteq N} \Delta_v(H)\, u_H$, where $\Delta_v(H)$ is the Harsanyi dividend of coalition H in the game v (Harsanyi, 1959). Now, the Shapley value is for every $i \in N$ defined by $\varphi_i(v) = \sum_{H \subseteq N \colon i \in H} \frac{\Delta_v(H)}{\# H}$. Hence, the Shapley value fairly distributes the generated Harsanyi dividends over the players that generate these dividends. The Shapley value was seminally introduced by Shapley (1953).

every $i \in N$ by

$$
\begin{aligned}
s_D(N) &= n_D \\
s_D(\{i\}) &= s_D(i) = \# D(i) \\
s_D(N-i) &= n_D - \# \left\{ j \in N_D^a \mid D^{-1}(j) = \{i\} \right\} = n_D - s_D^a(i)
\end{aligned}
$$

From this it follows that $M_i(s_D) = s_D(N) - s_D(N-i) = s_D^a(i)$ for every $i \in N$. Since $N_D^b \neq \varnothing$, this implies furthermore that $s_D(i) \geqslant M_i(s_D)$ for every $i \in N$. Therefore, the Gately value can be applied to this game.

From the previous we further derive that

$$
s_D(i) - M_i(s_D) = \# \left\{ j \in N \mid \{i\} \subsetneq D^{-1}(j) \right\} = s_D^b(i)
$$

and that

$$
\begin{aligned}
\sum_{j \in N} s_D(\{j\}) - s_D(N) &= \sum_{j \in N} s_D(j) - n_D = \sum_{h \in N} p_D(h) - n_D \\
&= \sum_{j \in N_D^b} (p_D(j) - 1) = \sum_{j \in N_D^b} p_D(j) - n_D^b.
\end{aligned}
$$

These properties imply that s_D is a regular TU-game as defined in Gilles and Mallozzi (2023). This implies that the Gately value applies to s_D.

We now compute the Gately value of the successor representation s_D. We note here that s_D is a concave cost game, implying that the reverse formulation of (6) needs to be applied. Hence, we derive for every $i \in N$ that

$$
\begin{aligned}
g_i(s_D) &= s_D(\{i\}) - \frac{s_D(\{i\}) - M_i(s_D)}{\sum_{j \in N} \left(s_D(\{i\}) - M_j(s_D) \right)} \cdot \left(\sum_{j \in N} s_D(\{j\}) - s_D(N) \right) \\
&= s_D(i) - \frac{s_D^b(i)}{\sum_{j \in N} s_D^b(j)} \cdot \left(\sum_{j \in N_D^b} p_D(j) - n_D^b \right) \\
&= s_D(i) - \frac{s_D^b(i)}{\sum_{j \in N_D^b} p_D(j)} \cdot \left(\sum_{j \in N_D^b} p_D(j) - n_D^b \right) \\
&= s_D(i) - s_D^b(i) + \frac{s_D^b(i)}{\sum_{j \in N_D^b} p_D(j)} \cdot n_D^b \\
&= s_D^a(i) + \frac{s_D^b(i)}{\sum_{j \in N_D^b} p_D(j)} \cdot n_D^b = \xi_i(D)
\end{aligned}
$$

Similarly, the conservative successor representation σ_D for the hierarchical network D is characterised for every $i \in N$ by

$$\begin{aligned}\sigma_D(N) &= n_D \\ \sigma_D(i) &= s_D^a(i) \\ \sigma_D(N-i) &= n_D - s_D(i)\end{aligned}$$

For the conservative successor representation σ_D we derive from the above that $M_i(\sigma_D) = s_D(i)$, implying that $\sigma_D(i) \leqslant M_i(\sigma_D)$ for every $i \in N$. Also, $\sigma_D(i) < M_i(\sigma_D)$ for some $i \in N$, since $N_D^b \neq \varnothing$. Therefore, σ_D is regular as defined in Gilles and Mallozzi (2023). Hence, the Gately value applies to σ_D.

Since σ_D is a convex game, the formulation stated in (6) applies. Now, we compute that

$$M_i(\sigma_D) - \sigma_D(i) = s_D(i) - s_D^a(i) = s_D^b(i)$$

and

$$\begin{aligned} g_i(\sigma_D) &= \sigma_D(\{i\}) + \frac{M_i(\sigma_D) - \sigma_D(\{i\})}{\sum_{j \in N}\left(M_j(\sigma_D) - \sigma_D(\{j\})\right)} \cdot \left(\sigma_D(N) - \sum_{j \in N} \sigma_D(\{j\})\right) \\ &= s_D^a(i) + \frac{s_D^b(i)}{\sum_{j \in N} s_D^b(j)} \cdot n_D^b = s_D^a(i) + \frac{s_D^b(i)}{\sum_{j \in N_D^b} p_D(j)} \cdot n_D^b = \xi_i(D) \end{aligned}$$

This shows the first equality in the assertion of the proposition.

Next, let $D \in \mathbb{D}^N$ be such that $N_D^b = \varnothing$. Then $p_D(j) = 1$ for all $j \in N_D$. This implies that for every $i \in N$: $M_i(s_D) = s_D(\{i\}) = s_D(i)$. Hence, for $i \in N$:

$$g_i(s_D) = s_D(\{i\}) = s_D(i) = s_D^a(i) = \xi_i(D).$$

Finally, for every $i \in N$: $M_i(\sigma_D) = s_D(i) = s_D^b(i) = 0$. Hence,

$$g_i(\sigma_D) = \sigma_D(\{i\}) = s_D^a(i) = \xi_i(D).$$

Combined with the previous case, this shows the first assertion of the proposition.

Finally, the second assertion of the proposition follows immediately from identifying the propensity to disrupt (Gilles and Mallozzi, 2023, Definition 3.2) in the successor representation s_D for some game theoretic imputation $m \in \mathbb{R}^N$ as

$$\frac{M_i(s_D) - s_D(\{i\})}{m_i - s_D(\{i\})} = \frac{s_D^b(i) - s_D(i)}{m_i - s_D(i)} = \frac{s_D^b(i)}{m_i - s_D(i)}.$$

Using the definition of a Gately point (Gilles and Mallozzi, 2023, Definition 3.2) and noting that $\xi_i(D) = s_D(i) = 0$ for every $i \in N_D^o$, the second assertion of the theorem is confirmed. ■

The Gately power measure introduced in Theorem 3.1 is founded on fundamentally different principles than the β-measure or other power measures. Now, the Gately power measure is member of a family of values that considers the control exercise over the nodes in N_D^b to be a collective resource in any hierarchical network $D \in \mathbb{D}^N$. The control is then allocated according to some well-chosen principle. In particular, The Gately measure allocates the control over N_D^b *proportionally* to the predecessor of the nodes in N_D^b. Assuming $N_D^b \neq \varnothing$, this proportional allocator is for every node $i \in N$ with $D(i) \cap N_D^b \neq \varnothing$ defined as

$$a_i(D) = \frac{s_D^b(i)}{\sum_{j \in N} s_D^b(j)} = \frac{s_D^b(i)}{\sum_{h \in N_D^b} p_D(h)} \tag{10}$$

where the Gately measure is now given by $\xi_i(D) = s_D^a(i) + a_i(D) \cdot n_D^b$.

This compares, for example, to the allocation principle based on the egalitarian allocator of the power over the nodes in $N_D^b \neq \varnothing$ given by

$$e_i(D) = \frac{1}{\# \{ j \in N \mid D(i) \cap N_D^b \neq \varnothing \}} \tag{11}$$

and the resulting *Restricted Egalitarian* power measure given by $\varepsilon_i(D) = s_D^a(i) + e_i(D)\, n_D^b$. We emphasise that the Gately and Restricted Egalitarian power measures are members of the same family of power measures for hierarchical networks, which have a collective allocative perspective on the control over the nodes in N_D^b.

3.1 Properties of the Gately measure

We investigate the properties of the Gately measure from the cooperative game theoretic perspective developed in this paper. We first investigate whether the Gately measure assigns a Core power gauge as is the case for the β-measure. Second, we consider some characterisations of the Gately measure. In particular, we derive an axiomatisation as well as investigate some interesting properties of the Gately measure on some special subclasses of hierarchical networks.

The Gately measure is not necessarily a Core power gauge We first establish that contrary to the property that the β-measure is the geometric centre of the set of Core power gauges of a given hierarchical network, its Gately power gauge does not necessarily have to satisfy the Core constraints. The next example provides a hierarchical network on a node set of 8 nodes which Gately measure is not a Core power gauge.

Example 3.2 Consider the hierarchical network D depicted in Figure 1 based on a node set $N = \{1, \dots, 8\}$. We note that $N_D = N_D^b = \{6, 7, 8\}$ and that nodes 1 and 2 fully control node 6, while nodes 3, 4 and 5 fully control nodes 7 and 8.

We compute that $s_D^a(i) = 0$ for all $i \in N$ and that, therefore, any power measure only considers the control arrangements of the nodes in N_D. In particular,

$$\beta(D) = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, 0, 0, 0 \right) \in C(D)$$

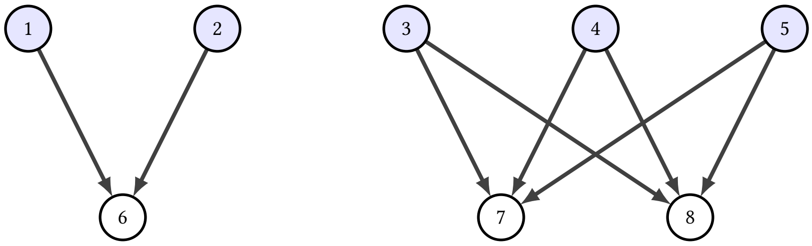

Figure 1: The hierarchical network considered in Example 3.2.

and that

$$\xi(D) = \left(\tfrac{3}{8}, \tfrac{3}{8}, \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}, 0, 0, 0 \right) \notin C(D).$$

Therefore, $\xi_1(D) + \xi_2(D) = \frac{3}{4} < 1 = \sigma_D(\{1,2\})$ shows that the Gately measure does not allocate sufficient power to the first two agents. The underlying reason is that the Gately value considers the control of *all* nodes in N^b_D to be a collective resource that is proportionally distributed according to $s^b_D(i)$. The relative low values of $s^b_D(1) = s^b_D(2) = 1$ in comparison with $s^b_D(3) = s^b_D(4) = s^b_D(5) = 2$ imply that the assigned share of the first two nodes is less than the total of fully controlled nodes by that node pair. ♦

The following theorem establishes under which conditions the Gately measure assigns a Core power gauge to a hierarchical network. The class of networks identified in (ii) compares with the classes of networks for which the Gately measure is identical to the β-measure.

Theorem 3.3 *Let $D \in \mathbb{D}^N$ be a hierarchical network on N.*

(i) *If $\#\{i \in N \mid D(i) \neq \varnothing\} \leqslant 3$, then $\xi(D) \in C(D)$.*

(ii) *If D is weakly regular, i.e., for all $i, j \in N^b_D$: $p_D(i) = p_D(j)$, then $\xi(D) \in C(D)$.*

Proof. Assertion (i) follows directly from Theorem 4.2 of Gilles and Mallozzi (2023). Here we note that the strong successor representation σ_D is essentially a three-player if $\#\{i \in N \mid D(i) \neq \varnothing\} = 3$ and a two-player game if $\#\{i \in N \mid D(i) \neq \varnothing\} = 2$. Both cases are covered by Theorem 4.2 of Gilles and Mallozzi, establishing that $\xi(D) = g(\sigma_D) \in C(\sigma_D) = C(D)$ as desired.

Let $p \geqslant 2$ such that $p_D(j) = p$ for all $j \in N^b_D$ as assumed. From this it follows that for $i \in N$:

$$\xi_i(D) = s^a_D(i) + \frac{n^b_D}{\sum_{j \in N^b_D} p_D(j)}\, s^b_D(i) = s^a_D(i) + \frac{n^b_D}{p \cdot n^b_D}\, s^b_D(i) = s^a_D(i) + \tfrac{1}{p}\, s^b_D(i)$$

Now let $H \subseteq N$ be some node group. Define

$$K_H = \left\{ j \in N^b_D \mid D^{-1}(j) \subseteq H \right\} \qquad \text{and} \qquad k_H = \#K_H \leqslant n^b_D.$$

Now note that

$$\sigma_d(H) = \#\left\{ j \in N \mid D^{-1}(j) \subseteq H \right\} = \sum_{i \in H} s^a_D(i) + k_H.$$

We next compute

$$\begin{aligned}
\sum_{i \in H} \xi_i(D) &= \sum_{i \in H} s^a_D(i) + \tfrac{1}{p} \sum_{i \in H} s^b_D(i) \\
&= \sum_{i \in H} s^a_D(i) + \tfrac{1}{p} \sum_{i \in H} \left[\#\{ j \in K_H \mid j \in D(i) \} + \# \left\{ j \in N^b_D \setminus K_H \mid j \in D(i) \right\} \right] \\
&\geqslant \sum_{i \in H} s^a_D(i) + \tfrac{1}{p} \sum_{i \in H} \#\{ j \in K_H \mid j \in D(i) \} \\
&= \sum_{i \in H} s^a_D(i) + \tfrac{1}{p} \sum_{j \in K_H} p_D(j) = \sum_{i \in H} s^a_D(i) + \tfrac{1}{p} \cdot p\, K_H \\
&= \sum_{i \in H} s^a_D(i) + K_H = \sigma_D(H).
\end{aligned}$$

Since H was arbitrary, this establishes that $\sum_{i \in H} \xi_i(D) \geqslant \sigma_D(H)$ for *all* coalitions $H \subseteq N$ and, therefore, $\xi(D) \in C(\sigma_D) = C(D)$, showing the second assertion of the theorem. ■

An axiomatic characterisation of the Gately measure We are able to devise a full axiomatisation of the Gately measure on $\mathbb{D}^N$ based on three defining properties. In order to state these properties, we define for every hierarchical network $D \in \mathbb{D}^N$ on node set N its *principal restriction* as the network $P_D \in \mathbb{D}^N$ defined by $P_D(i) = N^b_D \cap D(i)$ for every node $i \in N$. Similarly, a hierarchical network $D \in \mathbb{D}^N$ is a *principal network* if $D = P_D$. It is clear that a principal hierarchical network D is characterised by the property that $N^a_D = \varnothing$, meaning that all nodes with predecessors have actually multiple predecessors.

Theorem 3.4 *Let $\mathbb{D}^N$ be the class of hierarchical networks on node set N. Then the Gately measure $\xi \colon \mathbb{D}^N \to \mathbb{R}^N$ is the unique function $m \colon \mathbb{D}^N \to \mathbb{R}^N$ that satisfies the following three properties:*

(i) ***Normalisation:*** *m is n_D-normalised in the sense that $\sum_N m_i(D) = n_D$ for all $D \in \mathbb{D}^N$;*

(ii) ***Normality:*** *For every hierarchical network $D \in \mathbb{D}^N$ it holds that*

$$m(D) = s^a_D + m(P_D) \tag{12}$$

where $P_D \in \mathbb{D}^N$ is the principal restriction of D, and;

(iii) ***Restricted proportionality:*** *For every principal network $D \in \mathbb{D}^N$ with $D = P_D$ it holds that*

$$m(D) = \lambda_D\, s_D \qquad \text{for some } \lambda_D > 0. \tag{13}$$

Proof. We first show that the Gately measure ξ indeed satisfies these three properties. Normalisation trivially follows from the definition of the Gately measure. Let $D \in \mathbb{D}^N$ be an arbitrary hierarchical

network on N.
First, if $N^b_D = \varnothing$, we have that $s^b_D = 0 \in \mathbb{R}^N$ and $\xi(D) = s^a_D \in \mathbb{R}^N_+$. Hence, P_D is the empty network with $P_D(i) = \varnothing$ for all $i \in N$. Therefore, $s_{P_D} = s^b_D = 0$ and $\xi(P_D) = 0 = \lambda\, s^b_D = \lambda\, s_{P_D}$ for any $\lambda > 0$. This implies that $\xi(D)$ satisfies the normality property as well as restricted proportionality for the case that $N^b_D = \varnothing$.
Second, in the case that $N^b_D \neq \varnothing$, we have that

$$\xi(D) = s^a_D + \frac{n^b_D}{\sum_{j \in N^b_D} p_D(j)}\, s^b_D$$

Furthermore, we compute that for every $i \in N$: $\xi_i(P_D) = \frac{n^b_D}{\sum_{j \in N^b_D} p_D(j)}\, s^b_D(i) = \lambda_D\, s^b_D(i)$, where $\lambda_D = \frac{n^b_D}{\sum_{j \in N^b_D} p_D(j)}$. This implies that g satisfies restricted proportionality.
Next, let $m \colon \mathbb{D}^N \to \mathbb{R}^N$ be a power measure that satisfies the three given properties.
First, consider a network $D \in \mathbb{D}^N$ with $N^b_D = \varnothing$. Hence, $s^b_D = 0$, implying with (iii) that $m(P_D) = \lambda_D s^b_D = 0$. Therefore, with (ii), it follows that $m(D) = s^a_D = \xi(D)$.
Next, consider $D \in \mathbb{D}^N$ with $N^b_D \neq \varnothing$. Then, noting that $s_{P_D} = s^b_D$, from (iii) it follows that $m(P_D) = \lambda_D\, s^b_D > 0$[8] and with (ii) this implies that

$$m(D) = s^a_D + m(P_D) = s^a_D + \lambda_D\, s^b_D > 0.$$

Using the normalisation of m stated as property (i), we conclude that $\lambda_D = \frac{n^b_D}{\sum_{j \in N^b_D} p_D(j)}$ and, therefore, $m(D) = \xi(D)$.

Uniqueness of g as a power measure that satisfies the three listed properties is immediate and stated here without proof. ■

The three properties stated in Theorem 3.4 have a natural and direct interpretation. In particular, the normality property imposes that the power measure always assigns its uniquely subordinated nodes are assigned to a given node and that the main task of a power measure is to assign a power gauge for the principal restriction of any hierarchical network. This seems a rather natural hypothesis that is satisfied by other power measures such as the β-measure.

Restricted proportionality imposes that in a principal network the assigned power gauge is proportional to the number of other nodes that are controlled by that node. Again this seems a plausible hypothesis, even though it is violated by the β-measure.

In fact, the three properties are non-redundant as the following simple examples show:

- As indicated above, with regard to the axiomatisation devised in in Theorem 3.4, the β-measure satisfies the normalisation property (i) as well as the normality property (ii), but not the restricted proportionality property (iii). The Restricted Egalitarian power measure e is another example of a power measure on $\mathbb{D}^N$ that satisfies (I) as well as (ii), but not the Restricted Proportionality property (iii).

[8]We use the definition that for $x, y \in \mathbb{R}^N : x > y$ if and only if $x_i \geqslant y_i$ for all $i \in N$ and $x_j > y_j$ for some $j \in N$.

- Consider the proportional power measure ρ on $\mathbb{D}^N$ with for every $D \in \mathbb{D}^N$:

  $$\rho(D) = \frac{n_D}{\sum_{i \in N} s_D(i)}\, s_D.$$

  Then this proportional power measure satisfies the normalisation property (i) as well as the restricted proportionality property (iii), but not the normality property (ii) stated in Theorem 3.4.

- Finally, consider the direct power measure s on $\mathbb{D}^N$ with for every $D \in \mathbb{D}^N$:

  $$s(D) = s_D \in \mathbb{R}^N_+.$$

  This direct power measure s satisfies the restricted proportionality property (iiI) as well as the normality property (ii) stated in Theorem 3.4, but not the stated normalisation property (i).

3.2 A comparison between the β-measure and the Gately measure

On the class of weakly regular hierarchical networks, the Gately value satisfies the strong property that it coincides with the β-measure. This is explored in the next theorem.

Theorem 3.5 *Let $D \in \mathbb{D}^N_w$ be a weakly regular hierarchical network on N. Then the Gately measure coincides with the β-measure, i.e., $\xi(D) = \beta(D)$.*

Proof. To show the assertion, denote by $p = p_D(i) \geqslant 2$ the common number of predecessors for $i \in N^b_D$. Then it holds that

$$\sum_{j \in N} s^b_D(j) = \sum_{i \in N^b_D} p_D(i) = p \cdot n^b_D.$$

This implies simply that

$$\xi(D) = s^a_D + \frac{n^b_D}{p \cdot n^b_D}\, s^b_D = s^a_D + \tfrac{1}{p}\, s^b_D = \beta(D),$$

since for every $i \in N$: $\beta_i(D) = \sum_{j \in D(i)} \frac{1}{p_D(j)} = s^a_D(i) + s^b_D(i) \cdot \frac{1}{p}$. ■

Theorems 3.3 and 3.5 allow us to delineate networks with non-empty sets of Core power gauges that contain either the Gately measure, or the β-measure, or both, as well as determine when both of these measures coincide. This is explored in the next example.

Example 3.6 Consider the network D depicted in Figure 2 on the node set $N = \{1, \ldots, 5\}$. Note that this network satisfies the conditions of Theorem 3.3(i), but not of Theorem 3.5. Hence, $\xi(D) \in C(D) \neq \varnothing$, but $\xi(D) \neq \beta(D) \in C(D)$.

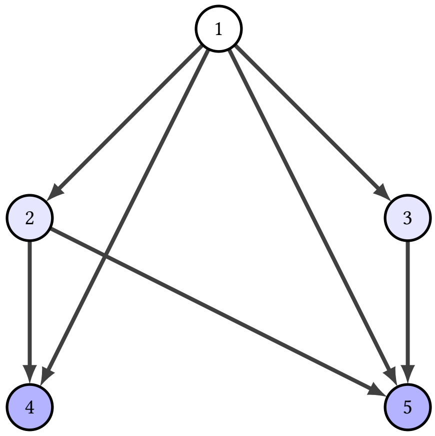

Figure 2: Network for Example 3.6.

We compute that the set of Core power gauges for this network is given by[9]

$$C(D) = \text{Conv}\ \{\ (2,1,1,0,0)\ ,\ (3,0,1,0,0)\ ,\ (3,1,0,0,0)\ ,\ (2,2,0,0,0)\ ,\ (4,0,0,0,0)\ \}$$

Next, we determine that the β-measure is in the (weighted) centre of $C(D)$ with

$$\beta(D) = \left(2\tfrac{5}{6}\ ,\ \tfrac{5}{6}\ ,\ \tfrac{1}{3}\ ,\ 0\ ,\ 0\right) \in C(D)$$

and that the Gately measure is computed as

$$\xi(D) = \left(2\tfrac{4}{5}\ ,\ \tfrac{4}{5}\ ,\ \tfrac{2}{5}\ ,\ 0\ ,\ 0\right) \in C(D).$$

Clearly, we have established that in this case $\xi(D) \neq \beta(D)$ even though the network D satisfies the condition of Theorem 3.3(i), implying that the Gately measure is a Core power gauge. ♦

A question remaining is whether the assertion of Theorem 3.5 can be reversed, i.e., if $\xi(D) = \beta(D)$ then D has to be weakly regular. The answer to that is negative as the following example shows.

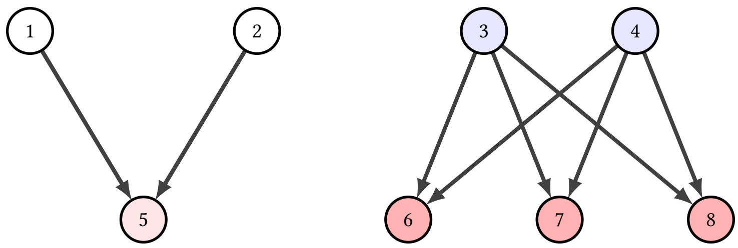

Figure 3: Network for Example 3.7.

[9]We point out that there are two simple subnetworks of D that result in exactly the same power gauge, namely the subnetwork in which node 1 dominates node 4 and node 2 dominates node 5 and vice versa. The resulting power gauge is $(3,1,0,0,0)$.

Example 3.7 Consider the node set $N = \{1, \dots, 8\}$ and the network D depicted in Figure 3. As the colour code indicates, there are four groups of nodes in this network. Nodes 1 and 2 together dominate node 5, while nodes 3 and 4 together dominate nodes 6,7 and 8.

Clearly, this network is not weakly regular. On the other hand, we claim that $\xi(D) = \beta(D)$. Now, we compute that $N^a_D = \varnothing$, $N^b_D = \{5, 6, 7, 8\}$ and $n^b_D = 4$. Furthermore, $\sum_{i \in N^b_D} p_D(i) = 2 + 3 \cdot 2 = 8$. This implies now that for every $i \in N$:

$$\xi_i(D) = \frac{n^b_D}{\sum_{i \in N^b_D} p_D(i)} \, s^b_D(i) = \tfrac{4}{8} \, s^b_D(i) = \tfrac{1}{2} \, s^b_D(i)$$

resulting in $\xi(D) = \left(\frac{1}{2}, \frac{1}{2}, 1\frac{1}{2}, 1\frac{1}{2}, 0, 0, 0 \right)$ and that this coincides with $\beta(D)$. ♦

References

Allouch, N., A. Meca, and K. Polotskaya (2021): "The Bonacich Shapley centrality," Discussion paper 2106, University of Kent, School of Economics, Canterbury, UK.

Arrigo, F., P. Grindrod, D. J. Higham, and V. Noferini (2018): "Non-backtracking walk centrality for directed networks," *Journal of Complex Networks*, 6, 54–78.

Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006): "Who's who in networks. Wanted: The key player," *Econometrica*, 74, 1403–1417.

Barabási, A.-L. (2016): *Network Science*, Cambridge, UK: Cambridge University Press.

Bavelas, A. (1948): "A mathematical model for group structures," *Human Organization*, 7, 16–30.

Bonacich, P. F. (1987): "Power and Centrality: A Family of Measures," *American Journal of Sociology*, 92, 1170–1182.

Brin, S. and L. Page (1998): "The Anatomy of a Large-Scale Hypertext Web Search Engine," *Computer Networks*, 30, 107–117.

Charnes, A., J. Rousseau, and L. Seiford (1978): "Complements, mollifiers and the propensity to disrupt," *International Journal of Game Theory*, 7, 37–50.

del Pozo, M., C. Manuel, E. González-Arangüena, and G. Owen (2011): "Centrality in Directed Social Networks: A Game Theoretic Approach," *Social Networks*, 33, 191–200.

Gately, D. (1974): "Sharing the Gains from Regional Cooperation: A game theoretic application to planning investment in electric power," *International Economic Review*, 15, 195–206.

Gavilán, E. C., C. Manuel, and R. van den Brink (2023): "Directed communication in games with directed graphs," *TOP*, https://doi.org/10.1007/s11750-023-00654-8.

Gilles, R. P. (2010): *The Cooperative Game Theory of Networks and Hierarchies*, Theory and Decision Library, Berlin, Germany: Springer Verlag.

Gilles, R. P. and L. Mallozzi (2023): "Gately Values of Cooperative Games," *arXiv*, 2208.10189, https://doi.org/10.48550/arXiv.2208.10189.

Gómez, D., E. González-Arangüena, C. Manuel, G. Owen, M. del Pozo, and J. Tejada (2003): "Centrality and Power in Social Networks: A Game Theoretic Approach," *Mathematical Social Sciences*, 46, 27–54.

Harsanyi, J. (1959): "A bargaining model for the cooperative n-person game," in *Contributions to the Theory of Games*, Princeton, NJ: Princeton University Press, vol. IV, 325–355.

Huremovic, K. and F. Vega-Redondo (2016): "Production Networks," HAL Working Paper 01370725.

Littlechild, S. C. and K. G. Vaidya (1976): "The propensity to disrupt and the disruption nucleolus of a characteristic function," *International Journal of Game Theory*, 5, 151–161.

Newman, M. E. J. (2005): "A measure of betweenness centrality based on random walks," *Social Networks*, 27, 39–54.

——— (2010): *Networks: An Introduction*, Oxford, UK: Oxford University Press.

Redhead, D. and E. A. Power (2022): "Social hierarchies and social networks in humans," *Philosophical Transactions of the Royal Society B*, 377, 20200440.

Shapley, L. S. (1953): "A Value for *n*-Person Games," in *Contributions to the Theory of Games*, ed. by R. Luce and A. Tucker, Princeton, NJ: Princeton University Press, vol. II.

——— (1971): "Cores of Convex Games," *International Journal of Game Theory*, 1, 11–26.

Skibski, O., T. P. Michalak, and T. Rahwan (2018): "Axiomatic characterization of game-theoretic centrality," *Journal of Artificial Intelligence Research*, 62, 33–68.

Staudacher, J. and J. Anwander (2019): "Conditions for the uniqueness of the Gately point for cooperative games," *arXiv*, 1901.01485, https://doi.org/10.48550/arXiv.1901.01485.

Tarkowski, M. K., T. P. Michalak, T. Rahwan, and M. Woolbridge (2018): "Game-Theoretic Network Centrality: A Review," *arXiv*, 1801.00218, https://doi.org/10.48550/arXiv.1801.00218.

van den Brink, R. and P. Borm (2002): "Digraph Competitions and Cooperative Games," *Theory and Decision*, 53, 327–342.

van den Brink, R., P. Borm, R. Hendrickx, and G. Owen (2008): "Characterizations of the β- and the Degree Network Power Measure," *Theory and Decision*, 64, 519–536.

van den Brink, R. and R. P. Gilles (1994): "A Social Power Index for Hierarchically Structured Populations of Economic Agents," in *Imperfections and Behavior in Economic Organizations*, ed. by R. P. Gilles and P. H. M. Ruys, Boston, MA: Kluwer Academic Publishers, Theory and Decision Library, chap. 12, 279–318.

——— (2000): "Measuring Domination in Directed Networks," *Social Networks*, 22, 141–157.

——— (2003): "Ranking by Outdegree for Directed Graphs," *Discrete Mathematics*, 271, 261–270.

van den Brink, R. and A. Rusinowska (2022): "The degree measure as utility function over positions in graphs and digraphs," *European Journal of Operational Research*, 299, 1033–1044.

White, D. R. and S. P. Borgatti (1994): "Betweenness Centrality Measures for Directed Graphs," *Social Networks*, 16, 335–346.

Yang-Yu, L., J.-J. Slotine, and A.-L. Barabási (2012): "Control centrality and hierarchical structure in complex networks," *PLOS One*, 7, e44459.