# Structured Production Systems: Viability[*]

Robert P. Gilles[†] Marialaura Pesce[‡]

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**Abstract**

This paper introduces a novel framework for analysing equilibrium in structured production systems incorporating a static social division of labour by distinguishing between consumption goods traded in competitive markets and intermediate goods exchanged through bilateral relationships. We develop the concept of viability—the requirement that all producers earn positive incomes—as a foundational equilibrium prerequisite.

Our main theoretical contribution establishes that acyclic production systems—those without circular conversion processes among goods—are always viable, a condition that implies coherence. We characterise completely viable systems through input restrictions demonstrating that prohibiting consumption goods as inputs for other consumption goods is necessary for ensuring viable prices exist for all consumption good price vectors. The analysis reveals fundamental relationships between production system architectural design and economic sustainability.

The introduced framework bridges Leontief-Sraffa production theory with modern network economics while capturing institutional realities of contemporary production systems. This also results in a contribution of the literature on the existence of a positive output price system and the Hawkins-Simon condition.

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[†]Department of Economics, The Queen's University of Belfast, Riddel Hall, 185 Stranmillis Road, Belfast, BT9 5EE, UK. Email: r.gilles@qub.ac.uk

[‡]**Corresponding author.** Università di Napoli "Federico II", Dipartimento di Scienze Economiche e Statistiche and CSEF, Complesso Universitario Monte Sant'Angelo Via Cintia, 80126 Napoli, Italy. Email: marialaura.pesce@unina.it

## 1 Supply chains and structured production

Modern economies are characterised by complex production networks where goods flow through multiple stages of transformation before reaching final consumers. This paper introduces a novel framework for analysing equilibrium in such structured production systems, departing fundamentally from traditional approaches by distinguishing between two categories of economic goods: consumption goods traded in competitive markets and intermediate goods exchanged through bilateral relationships within production chains. This distinction, while intuitive from an empirical standpoint, has profound implications for equilibrium theory and requires a fundamental reconceptualisation of how prices emerge and sustain productive activity.

**Differentiation of economic goods** Our framework recognises that economic goods serve fundamentally different roles in production systems. Consumption goods represent final outputs brought to competitive markets where prices emerge through supply and demand. These markets operate according to standard competitive principles, with prices serving as signals that coordinate the decisions of numerous anonymous agents. In contrast, intermediate goods function as inputs within production chains, traded bilaterally between specific producers according to technical requirements rather than through anonymous market mechanisms.

This distinction challenges the unified treatment of goods in classical general equilibrium theory—set out by Arrow and Debreu (1954) and Debreu (1959)—and input-output analysis (Leontief, 1936, 1941). While Arrow-Debreu models typically abstract from production chains through a model based on production sets, and Leontief models emphasise technical coefficients without explicit price formation mechanisms, our approach captures the dual nature of modern production systems where market and non-market, network exchanges coexist. The separation reflects empirical realities where intermediate goods often involve relationship-specific investments, technical specifications, and quality requirements that preclude anonymous market exchange. Their prices emerge through bargaining between connected producers rather than competitive forces. This institutional reality necessitates a theoretical framework that accommodates both competitive pricing for final goods and negotiated pricing for intermediates.

**A three-stage equilibrium conception** The comprehensive analysis of equilibrium in structured production systems requires a carefully sequenced theoretical conception. This paper initiates a three-stage equilibrium concept, with each stage addressing increasingly complex aspects of price determination and equilibrium.

The first stage, developed in this paper, establishes the concept of viability—the requirement that all producers earn positive incomes at prevailing prices. Without viability, producers would exit the system, disrupting production chains and preventing the realisation of any equilibrium. We characterise the structural properties of production systems that guarantee the existence of viable price systems, establishing connections between the architecture or design of the production system and economic sustainability. This analysis relies on the use of matrix theory, in particular the Hawkins-Simon condition (Hawkins and Simon, 1949) and McKenzie's condition of a quasi-dominant diagonal in a $\mathcal{Z}$-matrix (McKenzie, 1957). This foundational analysis identifies when price systems can sustain all productive activities, a prerequisite for any meaningful equilibrium concept.

The second stage, to be developed in a companion paper, introduces a general equilibrium concept that combines viability with market clearing in consumption good markets. This equilibrium notion must reconcile two distinct coordination mechanisms: competitive markets for consumption goods and bilateral bargaining relationships for intermediate goods. The challenge lies in proving existence while respecting the institutional constraints that intermediate goods lack organised markets.

The third stage will endogenise intermediate good prices through explicit bargaining mechanisms. We envision two complementary approaches. First, reference groups of producers with similar economic standing may negotiate on equal footing, leading to income equalisation within groups. Alternatively, a Nash bargaining framework can model situations where producers possess differential bargaining power, yielding specific income distributions determined by network position and economic leverage.

## 1.1 Relationship to existing literature

**Production networks and structural analysis** Recent advances in network economics have renewed interest in production structures. Baqaee (2018) demonstrates how entry and exit in production networks create cascading failures and amplify shocks, challenging the notion that sales shares adequately capture systemic importance. Unlike the relevant notions of centrality in competitive models, systemic importance depends on an industry's role as both supplier and consumer of inputs, as well as market structure. Baqaee and Farhi (2019a) extend this analysis, showing that microeconomic details of production structure—network linkages, elasticities of substitution, returns to scale—shape second-order terms in aggregate responses to shocks, going beyond Hulten's theorem. Their work on the microeconomic foundations of aggregate production functions (Baqaee and Farhi, 2019b) establishes that aggregation from heterogeneous micro-level technologies to macro production functions depends critically on the economy's network structure and elasticities of substitution. These contributions, along with their analysis of productivity and misallocation in general equilibrium (Baqaee and Farhi, 2020), demonstrate that production network structure fundamentally affects the mapping from microeconomic shocks to aggregate outcomes across different frequencies.

The role of production networks in shaping macroeconomic dynamics extends to monetary policy. Huang and Liu (2001) analyse a multi-stage production economy, demonstrating that the input-output structure creates strategic complementarities in price-setting. Building on this insight, La'O and Tahbaz-Saheli (2022) characterize optimal monetary policy in production networks with nominal rigidities, showing that the optimal policy stabilises a price index with greater weights on larger, stickier, and more upstream industries. Their analysis reveals that network structure fundamentally shapes the transmission of monetary policy and the trade-offs faced by monetary authorities. Pellet and Tahbaz-Salehi (2023) further explore how quantity rigidities and informational frictions in production networks dampen the impact of productivity shocks while amplifying the effects of demand shocks, with the magnitude depending on the network's structure.

Levine (2012) emphasizes the "weakest link" property where long chains permit specialisation but create vulnerability to failure. This motivates our focus on viability: if any producer fails, downstream producers may also fail, creating cascading disruptions. Bosker et al. (2014) examines how production networks affect economic geography, showing that network structure interacts with trade costs to determine spatial patterns of production.

**Bargaining and intermediation** The literature on bargaining in networks provides important insights for understanding price formation for intermediate goods. Siedlarek (2025) develops a stochastic bargaining model for networked markets with intermediaries, showing that non-essential players—those who can be circumvented through alternative routes—receive zero payoffs as bargaining frictions vanish, while essential players extract positive rents. This result provides microfoundations for thinking about how network position affects bargaining power and price formation in production chains.

Acemoglu and Azar (2020) analyse endogenous production networks with Nash bargaining between firms, examining how bargaining power affects markups, prices, and profits throughout supply chains. They show that an increase in a firm's bargaining power vis-à-vis its customer propagates upstream, affecting all its direct and indirect suppliers. This demonstrates how bilateral bargaining shapes the distribution of surplus in production networks. Bizzarri (2023) provides a model of general equilibrium oligopoly in input-output networks where all firms have market power on both input and output markets and are fully strategic about their position in the supply chain.

**Frictions and trading networks** Fleiner et al. (2019, 2022) develop a theory of trading networks with distortionary frictions that make utility imperfectly transferable between agents. They establish existence of competitive equilibria under these conditions and provide cooperative foundations for competitive equilibrium in finite markets with frictions. Their work demonstrates that many structural results—including lattice properties and versions of the rural hospitals theorem (Roth and Sotomayor, 1990)—extend to environments with trading frictions. While their focus is on exchange networks rather than production networks, their results on how frictions affect equilibrium outcomes inform our analysis of price formation under institutional constraints. Huremovic et al. (2020) analyse how financial shocks propagate through production networks, showing that the interaction between financial frictions and production linkages amplifies aggregate volatility.

The supply chain management literature (Cachon, 2003) and work on vertical integration (Williamson, 1985) document how relationship-specific investments and contractual incompleteness shape production organisation. Game-theoretic approaches (Kranton and Minehart, 2001) explore how bilateral relationships affect economic outcomes. Our framework extends this literature by embedding bargaining within a general equilibrium structure.

**Our contribution**

Our framework distinguishes itself by treating consumption goods and intermediate goods as fundamentally different economic objects requiring distinct pricing mechanisms: competitive markets for the former and bilateral bargaining for the latter. By allowing consumption goods to have competitive prices while intermediate goods have negotiated prices, we capture essential features of real production systems obscured by traditional approaches.

The concept of viability introduced here provides a minimal requirement for economic sustainability, weaker than full equilibrium but essential for any stable configuration. Unlike existence results in Baqaee (2018) focusing on free entry equilibria with external economies of scale, or the general equilibrium frameworks of Baqaee and Farhi (2020) emphasising misallocation and productivity, our viability analysis asks the more fundamental question: which production network structures admit price systems that allow all producers to cover their costs? This is a necessary condition for any equilibrium, whether competitive, oligopolistic, or based on bargaining.

Our characterisation of viable production systems through structural properties—particularly acyclicity and input restrictions—offers tractable conditions for determining when price systems can support all productive activities. We show that acyclic production systems are always viable, extending insights from Levine (2012) on production chains to our setting with consumption and intermediate goods. While cyclic systems may still admit viable prices under certain conditions, we establish that the completely viable property—ensuring viable prices exist for all consumption good price vectors—emerges as a key requirement for robust production systems. This property requires prohibiting consumption goods as inputs for other consumption goods, suggesting that certain production architectures are inherently more stable than others.

The paper establishes fundamental relationships between production network structure and the existence of viable prices. These structural results complement the emphasis on network topology in Baqaee (2018) and Baqaee and Farhi (2019a), but focus on the static question of sustainability rather than dynamic responses to shocks. Our findings have implications for industrial organisation and economic development. By separating the analysis of viability from market clearing and bargaining, we provide foundations for a richer equilibrium theory that can accommodate the insights from Siedlarek (2025) on bargaining power and network position, from Acemoglu and Azar (2020) on supply chain negotiations, and from Fleiner et al. (2019) on frictions in trading networks. This modular approach allows us to build incrementally toward a comprehensive theory of equilibrium in structured production systems.

### 1.2 Organisation of the paper

The remainder of this paper develops the analysis of viability in structured production systems. Section 2 introduces the formal framework, defining structured production systems and their key properties. Section 3 analyses viability, establishing existence results and characterising the relationship between structural properties and viable prices. Section 4 examines complete viability and its connection to input restrictions. Section 5 concludes, outlining the broader research program and discussing implications for economic theory and policy.

## 2 Structured production systems

Our aim is to develop a mathematical model that captures a production system that is founded on structured or chained production processes through fully specialised economic agents. Hence, production processes are fully expressed in all input and output flows related to all individual production units and the accompanying trade between these units. The final outflows from the professional production system are made up of the produced quantities of consumables that are consumed by the population of economic agents in the production system. As such, the model represents a *structured* supply chain production system based on fully specialised production units, incorporating a static social division of labour.

Each production unit or *producer* in a production chain is represented as a single economic decision maker—an economic agent—and each agent in the production system embodies exactly one production unit in the professional production system. All economic agents are producers and assume some "professional" function in the production system. Hence, these producers are fully specialised, each assuming exactly one of a finite number of "professions", each represented by a fully specialised (fixed) production plan.

We assume throughout that each producer is fully specialised in the production of a *single* output. This output can be *either* a final consumption good, *or* an intermediate input for productive tasks of other producers. Consequently, in this model production is ordered through supply chains of fully specialised economic agents.

In these structured production processes, fully specialised producers trade intermediate goods among themselves. Their trade is solely determined by the technical requirements of the production processes, and these goods are never consumed. The prices of intermediate goods are determined through bargaining, making them at best *partially competitive*.

On the other hand, consumption goods are brought to market in the same way as in any market economy. The supply of these goods is confronted with the demand for them in a standard fashion. Consequently, consumption good prices are determined through the interaction of supply and demand in competitive markets.

**Mathematical concepts and notation**

Throughout the paper we use a number of mathematical concepts that we list here for convenience of the reader.

Throughout we use the following convention for vector notation. Let $x, y \in \mathbb{R}^K$ be two vectors in the standard $K$-dimensional Euclidean space. Then we denote $x \geqslant y$ if $x_k \geqslant y_k$ for all $k = 1, \ldots, K$. Next, we say that $x > y$ if $x \geqslant y$ and $x \neq y$. Finally, we denote by $x \gg y$ that $x_k > y_k$ for all $k = 1, \ldots, K$.

The open interior of a set $X \subset \mathbb{R}^K$ is denoted as $\operatorname{int} X \subseteq X \subset \mathbb{R}^K$. Furthermore, if a set $X \subset \mathbb{R}^K$ is contained in a lower-dimensional subspace $Y$ of $\mathbb{R}^K$, then its relative interior, denoted by $\operatorname{ri} X \subset Y$, is the interior of $X$ relative to the subspace $Y$.

A *polytope* in $\mathbb{R}^K$ is a set that is the convex hull of a finite set of vectors or points in $\mathbb{R}^K$. Alternatively, by the Minkowski-Weyl Theorem, a polytope can be understood as a bounded set that is the intersection of a finite number of closed half-spaces, $H(n, c) = \{x \in \mathbb{R}^K \mid n \cdot x \geqslant c\}$ for some norm $n \in \mathbb{R}^K \setminus \{0\}$ and $c \in \mathbb{R}$, see Balestro et al. (2024, Proposition 3.1.2)—and Rockafellar (1970, Theorem 19.1) for the more general class of polyhedra.

We denote by $0 \in \mathbb{R}^K$ the vector of all zeroes. A vector $x \in \mathbb{R}^K$ is non-negative (non-positive) if and only if $x \geqslant 0$ ($x \leqslant 0$); $x \in \mathbb{R}^K$ is negative (positive) if and only if $x < 0$ ($x > 0$); and, $x \in \mathbb{R}^K$ is strictly negative (strictly positive) if and only if $x \ll 0$ ($x \gg 0$).

A square matrix $A = (a_{ij})$ is a $\mathcal{Z}$-matrix or a matrix of class $\mathcal{Z}$ if $a_{ij} \leqslant 0$ for any $i \neq j$. A $\mathcal{Z}$-matrix $A = (a_{ij})$ is of class $\mathcal{Z}^+$ if it has a positive (main) diagonal, i.e., if $a_{ii} > 0$ for any $i$ (Giorgi, 2022).

A $\mathcal{Z}$-matrix $A$ satisfies the Hawkins-Simon condition (Hawkins and Simon, 1949) if the leading principal minors of $A$ are all positive. McKenzie (1960) extended the notion of matrices with *dominant diagonal*, namely developed by Hadamard (1903) as follows:

**Definition 2.1** *A square real matrix $A = (a_{ij})$ of order $K \in \mathbb{N}$ is said to have a* ***row quasi-dominant diagonal*** *(a* ***column quasi-dominant diagonal****) if there exist positive number $d_1, d_2, \dots, d_K > 0$ such that*

$$d_i|a_{ii}| > \sum_{j \neq i} d_j |a_{ij}| \quad \text{for all } i = 1, \dots, K \qquad \left( d_j|a_{jj}| > \sum_{i \neq j} d_i |a_{ij}| \quad \text{for all } j = 1, \dots, K \right)$$

Unlike the notion of dominant diagonal, if $A$ has a row quasi-dominant diagonal, then it has a column quasi-dominant diagonal and vice-versa. Therefore, we simply refer to these matrices as having a *quasi-dominant diagonal* (q.d.d.). If, in addition, $a_{ii} > 0$ for all $i = 1, \dots, K$, then $A$ is said to have a *positive quasi-dominant diagonal* (p.q.d.d.). McKenzie (1960) shows that:

(i) If $A$ has a q.d.d., then $A$ is nonsingular;

(ii) If $A$ has a p.q.d.d., then all principal minors of $A$ are positive.

Theorems 1 and 4 of Giorgi (2023) summarise the state of the literature on investigating equivalent statements of the Hawkins-Simon condition.

**Lemma 2.2** *Let $A = (a_{ij})$ be a square matrix of order $K \in \mathbb{N}$ that is of class $\mathcal{Z}^+$. Then the following statements are equivalent:*

(a) *There exists some $x \in \mathbb{R}^K : x > 0$ and $Ax \gg 0$;*

(b) *For any $y \in \mathbb{R}^K$ with $y \geqslant 0$, there exists a $x \in \mathbb{R}^K_+$ such that $Ax = y$;*

(c) (Hawkins-Simon condition) *All leading principal minors of $A$ are positive;*

(d) *All principal minors of $A$ are positive;*

(e) *The matrix $A$ has a positive quasi-dominant diagonal (p.q.d.d.).*

## 2.1 Static intermediated production structures

In this section we develop a mathematical framework that captures the principles laid out in the introduction. Principally, our approach is based on the distinction between final consumption goods and intermediate goods. This distinction, exemplified in using separate representations of these two types of goods, is central in our model. The model introduced here was initially set out by Gilles (2019, Chapter 6).

**Production of consumables and intermediate goods** There are $\ell_c \in \mathbb{N}$ consumption goods, which are produced through processes founded on the use of $\ell_p \geqslant 0$ intermediate goods or products. Consumption goods are considered to be final outputs of the production system and brought to market.

We introduce $L_c = \{1, \ldots, \ell_c\}$ as the corresponding set of consumption goods. These goods are traded in $\ell_c$ independent competitive markets. Throughout we assume that $\ell_c \geqslant 1$, which guarantees a meaningful production framework generating at least one consumptive final output.

On the other hand, intermediate goods or inputs are trucked and bartered in bargaining processes between production units in the supply chains. The set of intermediate goods is $L_p = \{1, \ldots, \ell_p\}$ if $\ell_p \in \mathbb{N}$ and $L_p = \varnothing$ if $\ell_p = 0$.[1]

Within this setting the consumption space is determined as the non-negative orthant of the $\ell_c$-dimensional Euclidean space $\mathbb{R}^{\ell_c}_+$. On the other hand, all production processes take place within the total commodity space $\mathbb{R}^\ell$, where $\ell = \ell_c + \ell_p$ is the number of all goods in the production system. We assume that all goods in $L$, where $L = L_c \cup L_p$, are tradable in the production system, or in the consumption good markets.

We assume that each good $k \in L$ is produced by producers who assume the corresponding profession related to producing good $k$. The profession for producing good $k$ is fixed and determined fully by the production technology for that particular good. Therefore, there are exactly $\ell$ professions and $L$ can also be interpreted as the set of professions in the structured production system.

We assume that the production of each good $k \in L$ is efficient: Every agent who assumes the profession related to good $k$ is assumed to produce exactly according to the same "professional" production plan.[2] In particular, a fully specialised professional producer of good $k$ has a given output of $Q^k > 0$ of that good, which requires the input of a bundle $y^k \in \mathbb{R}^\ell_+$. This is formalised next.

**Definition 2.3** *For each good $k \in L$, the corresponding **profession** is defined by a fully specialised production plan with good $k$ being the only output. This can be represented as*

$$\zeta(k) = Q^k\, e^k - y^k \in \mathbb{R}^\ell \tag{1}$$

*where $e^k$ is the $k$-th standard Euclidean unit vector in the commodity space $\mathbb{R}^\ell$, $Q^k > 0$ is the fixed generated output quantity of good $k$, and $y^k \in \mathbb{R}^\ell_+$ with $y^k_k = 0$ is the required input vector to generate $Q^k$ units of good $k$.*
*The mapping $\zeta \colon L \to \mathbb{R}^\ell$ with $\zeta(k) = Q^k\, e^k - y^k$ now represents the corresponding complete **professional production system**.*

We note that the input vector $y^k$ might require positive input quantities of consumables as well as intermediate goods. From this formalisation we can identify that good $m \in L$ is an *intermediate input for the production of* $k \in L$ if $y^k_m > 0$, where $y^k_m$ is the amount of good $m$ that is used in the

[1] In particular, the property that $\ell_p = 0$ refers to the case that all consumption goods are home-produced which is the foundation of a pure exchange economy (Arrow and Debreu, 1954; Debreu, 1959). Indeed, in this case, every production plan is actually equivalent to an initial endowment.

[2] This corresponds to an implementation of the hypothesis of Marshallian competition between producers (Marshall, 1890). The standards of the profession are productively efficient in the sense that they generate the highest surplus per produced unit. Any deviation from these standards would result in lower surpluses and reduced generated incomes.

production of $Q^k$ of good $k$. Definition 2.3 imposes that $\zeta(k) \in \mathbb{R}^\ell$ is a vector in which only the $k$-th coordinate has a positive entry, while all other entries are non-positive. In particular, good $k \in L$ is never an intermediate input for its own production, since $y^k_k = 0$.

If $y^k = 0$, we refer to profession $k \in L$ as being a home production technology, since it requires no intermediate inputs into its production process. This refers to the case that good $k$ is produced through a *home production* technology, i.e., $\zeta(k) = Q^k e_k$ with $e_k$ being the $k$-th unit vector in the commodity space $\mathbb{R}^\ell$.

**Matrix representation of $\zeta$** It is useful to use a matrix representation of a professional production system $\zeta$. This matrix representation lists all professions and their constituting production plans in an $(\ell \times \ell)$ matrix.

Formally, denote by $Z$ the $(\ell \times \ell)$ matrix whose rows are defined by $\zeta(k)^\top = (Q^k e_k - y^k)^\top$ with $e_k$ the $k$-th unit vector for every $k \in L$. Hence, $Z = (z_{ij})_{i,j \in L}$ where $z_{ii} = Q^i > 0$ for every $i \in L$ and $z_{ij} = -y^i_j \leqslant 0$ for all $i, j \in L$ with $i \neq j$. Thus,

$$Z = \begin{bmatrix} Q^1 & -y^1_2 & \cdots & y^1_\ell \\ -y^2_1 & Q^2 & \cdots & y^2_\ell \\ \vdots & \vdots & \ddots & \vdots \\ -y^\ell_1 & -y^\ell_2 & \cdots & Q^\ell \end{bmatrix} \tag{2}$$

It is clear from the construction of $Z$ that $Z$ is actually a matrix of class $\mathcal{Z}^+$.

Throughout we use the convention that all consumption good production plans $k \in L_c$ are listed in the first $\ell_c$ rows of the matrix $Z$. All intermediate good production plans $m \in L_p$ are listed in rows $\ell_c + 1, \ldots, \ell$ of the matrix $Z$, noting that $\ell = \ell_c + \ell_p$.

**Acyclic production** We formalise next how the professional production system $\zeta$ on $\mathbb{R}^\ell$ exhibits properties that all production is meaningful and results in the targeted, effective production of the final consumption goods that drive wealth generation in such a system. The next formalisation of acyclicity captures the idea that production processes are purposeful and lead most effectively to the production of the resulting consumptive outputs.

**Definition 2.4** *Let $\zeta : L \to \mathbb{R}^\ell$ be a professional production system.*

(a) *A **cycle** in the professional production system $\zeta$ is an ordered subset $\{k_1, \ldots, k_q\} \subseteq L$ with $q \in \mathbb{N} \setminus \{1\}$ such that*

  (i) *$k_q$ is an input in the production of $k_1$, i.e., $y^{k_1}_{k_q} > 0$, and*

  (ii) *for every $i = 1, \ldots, q-1$ it holds that $k_i$ is an input in the production of $k_{i+1}$, i.e., $y^{k_{i+1}}_{k_i} > 0$.*

(b) *The professional production system $\zeta$ is **acyclic** if it does not admit any cycles.*

Acyclicity of a professional production system $\zeta$ is a rather strong consistency and efficiency property. It excludes any cycle to be possible in any implementation of a production system by assigning economic agents to the professions represented in $\zeta$.

Acyclicity of a professional production system is a structural efficiency property that imposes that all production is purposeful, resulting into production processes leading systematically to the output of consumption goods only. There are no production cycles in which intermediate goods are converted into other intermediate goods; all production is "purposeful" in the sense that there are no backward good flows in the consumption good production.

It is straightforward to establish the next reformulation of the acyclicity property. It rephrases the absence of productive input cycles by stipulating that all multiplications of cross-inputs are zero.

**Remark 2.5** *The professional production system* $\zeta \colon L \longrightarrow \mathbb{R}^{\ell}$ *is acyclic if and only if for every ordered subset* $\{k_1, \ldots, k_q\} \subseteq L$ *with* $q \in \mathbb{N} \setminus \{1\}$ *it holds that*

$$y^{k_1}_{k_q} \cdot \prod_{i=1}^{q-1} y^{k_{i+1}}_{k_i} = 0 \tag{3}$$

The definition and meaning of acyclicity is illustrated in the next example.

**Example 2.6** Consider a professional production system $\zeta$ with one intermediate good $I$ and two consumables, $X$ and $Y$. The intermediate input is produced through the production plan $\zeta(I) = (0, -1, 2)$, using $Y$ as an input. Consumable $X$ is produced with a two units of the intermediate input, represented by the production plan $\zeta(X) = (x, 0, -2)$, where $x > 3$. Finally, consumable $Y$ is produced based on a three units of the consumable $X$, resulting in the corresponding production plan $\zeta(Y) = (-3, y, 0)$ with $y > 1$.
The $(3 \times 3)$ matrix representation of $\zeta$ as defined by (2) can be given as

$$Z = \begin{bmatrix} x & 0 & -2 \\ -3 & y & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

There is a cycle in this simple professional production system. Indeed, the ordered set $\{I, X, Y\}$ defines a cycle with $y^X_I = 2 > 0$, $y^Y_X = 3 > 0$ and $y^I_Y = 1 > 0$. Hence, $y^X_I \cdot y^Y_X \cdot y^I_Y = 2 \cdot 3 \cdot 1 = 6 \neq 0$. With Remark 2.5 we confirm that $\zeta$ is not acyclic. ♦

## 2.2 Definition of structured production systems

Throughout we assume that the set of agents in the production system is given by $N = \{1, \ldots, n\}$. Our framework demands $n \geqslant \ell = \ell_c + \ell_p$, ensuring that all goods can be produced by assigning professions in the professional production system $\zeta$ to the agents in $N$. Consequently, each introduced profession $k \in L$ is assumed by at least one, but potentially multiple, agents in $N$.

The following definition introduces the main concept that underpins the analysis presented here. We understand a structured production system to be based on the production of goods through objective professions. The outputs of these professions are either traded in a competitive market (for consumption goods) or traded along a binary relationship between producers (for intermediate goods).

**Definition 2.7** *A* ***structured production system*** *with $\ell_c \geqslant 2$ consumables, $\ell_p \geqslant 0$ intermediary goods and $n \geqslant \ell = \ell_c + \ell_p$ economic agents is a tuple $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ where*

- *$N = \{1, \ldots, n\}$ is a set of economic agents, represented as fully specialised producers,*
- *$\zeta \colon L \longrightarrow \mathbb{R}^{\ell}$ is a professional production system, describing the profession-based production technology in the system as introduced in Definition 2.3, and*
- *$\gamma \colon N \longrightarrow L$ is an assignment of professions to all economic agents in the production system,*

*such that there exists some $e \in \mathbb{R}^{\ell_c}_{++}$ with*

$$\sum_{a \in N} \zeta(\gamma(a)) = (e, 0) \in \mathbb{R}^{\ell}. \tag{4}$$

*A structured production system $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ is* ***acyclic*** *if its professional production system $\zeta$ is acyclic in the sense of Definition 2.4.*

The property stated in (4) implies that all intermediate goods generated within a production system are absorbed into overall production processes. Consequently, the net output of the production system is represented by a strictly positive vector of consumption goods, denoted as $e \gg 0$.

Property (4), therefore, is a productive efficiency hypothesis. All intermediate inputs are produced in quantities that *exactly* suffice for the production of the consumable goods.

Furthermore, Property (4) allows for a meaningful confrontation between the generated quantities of *all* consumption goods and the generated demand from consumers, as the total output of the professional production system in the production system consists solely of consumption goods.

It should be clear that a structured production system represents production that corresponds to a specific implementation of Leontief's input-output framework (Leontief, 1936, 1941). Instead of industrial sectors, we use professions assigned to individual producers in the description of production. Additionally, we modify the input-output framework with a description of consumptive activities, enabling an analysis of equilibrium pricing processes.

**Comparison with network representation of production systems** Gilles (2019, Chapter 6) introduces economies with production networks, further explored in Gilles and Pesce (2025a). We mention here that every production network in the sense of Gilles and Pesce (2025a) is actually a specific implementation of a structured production system as defined above. Indeed, the positions in a production network are occupied by fully specialised economic agents, each being assigned a profession defined in Definition 2.3. Links in a production network are weighted and describe a quantity of a commodity traded between two economic agents. The in-flows and outflows of each position in the production network exactly add up to the production plan representing the profession of the economic agent constituting this position.

It should be clear that each production network implements a well-defined structured production system. Conversely, the $\mathcal{Z}$-matrix representation of some professional production system $\zeta$ itself is an adjacency matrix of some weighted network in only very specific circumstances. This refers to the special case in which it is possible that each profession can be assumed by a single "representative agent" to form a structured production system satisfying (4).

On the other hand, it should be clear that each production network contains more information than is contained in the corresponding structured production system. For details we refer to Gilles (2019, Chapter 6) and Gilles and Pesce (2025a).

**Coherent structured production systems** We introduced the notion of acyclicity of a professional production system $\zeta$ in Definition 2.4. In the context of a structured production system $\mathbb{S}$ we can introduce a weaker notion of coherence that imposes that there no meaningless cyclic conversion processes of intermediate goods only. This is formalised as follows.

**Definition 2.8** *Let $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ be some structured production system and let $N_k = \{i \in N \mid \gamma(i) = k\}$ be the class of agents assigned to profession $k \in L$.*
*A **conversion cycle** in $\mathbb{S}$ is a sub-population vector $n \in (\mathbb{N} \cup \{0\})^L$ with $n_k \leqslant \# N_k$ for every $k \in L$ such that $n \neq 0$ and*

$$n \cdot Z = \sum_{k \in L} n_k \, \zeta(k) = 0, \qquad \text{where } Z \text{ represents } \zeta. \tag{5}$$

*The structured production system $\mathbb{S}$ is **coherent** if it does not admit any conversion cycles.*

**Remark 2.9** *Note that the presence of a conversion cycle in structured production $\mathbb{S}$ implies that the production plans $\{\zeta(k) \mid k \in L\}$ are linearly dependent. Hence, $\mathbb{S}$ is coherent if and only if $\{\zeta(k) \mid k \in L\}$ are linearly independent if and only if the matrix $Z$ representing $\zeta$ is non-singular, i.e.,* $\det Z \neq 0$.

Coherence means that the structured production system does not contain non-meaningful production of intermediate goods. Therefore, all intermediate goods are directly or indirectly used in the production of consumption goods. This has consequences for the matrix representation of the underlying professional production system $\zeta$. Indeed, the matrix cannot have a block structure. This is illustrated in the next example.

**Example 2.10** Consider the production system $\mathbb{S}$ with three intermediate goods $A$, $B$ and $C$ and one consumable good $X$. The production of $X$ only depends on intermediate input $C$, while $A$ and $B$ are converted into each other. A corresponding professional production system $\zeta$ can be represented as

$$Z = \begin{bmatrix} x & 0 & 0 & -\gamma \\ 0 & \alpha & -\beta & 0 \\ 0 & -\alpha & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \qquad \text{with } \alpha,\ \beta,\ \gamma > 0.$$

Note that $Z$ has a $2 \times 2$ block that represents that $\alpha$ units of $A$ is converted in $\beta$ units of $B$, and vice-versa. This indicates that this structured production system is *not* coherent. Indeed, let $n = (0, 1, 1, 0)$, then $n \cdot Z = 0$, indicating that $\mathbb{S}$ contains a conversion cycle.

In particular, we compute that

$$\det Z = x \cdot \gamma \cdot \det \begin{bmatrix} \alpha & -\beta \\ -\alpha & \beta \end{bmatrix} = 0$$

This confirms by Remark 2.9 that $\mathbb{S}$ is not coherent. ♦

Conversion cycles are implemented social divisions of labour on a sub-population in a structured production system that has no proper output of any goods. It means that within this social division of labour there is conversion of intermediate goods into other intermediate goods without this resulting in the output of consumption goods, making this conversion meaningless.

We note that acyclicity of a professional production system $\zeta$ implies that there are no cycles in the system based on these production plans. Therefore, there will be no conversion cycles in any structured production system $\mathbb{S}$ founded on the professional production system $\zeta$.

**Remark 2.11** *If $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ is a structured production system such that the professional production system $\zeta$ is acyclic, then $\mathbb{S}$ is coherent.*
*The converse is not true. Indeed, the professional production system $\zeta$ considered in Example 2.6 is not acyclic, but any structured production system based on this professional production system is coherent as $\det Z = 2xy - 6 > 0$, since $x > 3$ and $y > 1$.*

## 3 Viability of structured production systems

Next, we investigate how generated wealth can be distributed throughout a structured production system. We base ourselves on the established principle that wealth distribution is effectuated through the pricing of goods.

Consider a structured production system $\mathbb{S} = \langle N, \zeta, \gamma \rangle$. Now, a *price system* in $\mathbb{S}$ is denoted as a $\ell$-dimensional vector $(p, q) \in \mathbb{R}^{\ell}_{+}$ with $p \neq 0$ where the price vector of consumption goods $p \neq 0$ is normalised in the standard fashion[3] as

$$p \in \bar{S} = \left\{ (p_1, \dots, p_{\ell_c}) \,\middle|\, p_k \geqslant 0 \text{ for all } k \in L_c \text{ and } \sum_{k \in L_c} p_k = 1 \right\}$$

and $q \in \mathbb{R}^{\ell_p}_+$ is a vector of intermediate good prices.

The consumption good price space $\bar{S}$ is the unit simplex in $\mathbb{R}^{\ell_c}$ and consists of positive normalised consumption good price vectors $p > 0$ only. On the other hand, the intermediate price space $\mathbb{R}^{\ell_p}_+$ is the non-negative orthant of the $\ell_p$-dimensional Euclidean space $\mathbb{R}^{\ell_p}$. As such, the intermediate goods price vectors $q$ are *not* normalised, but can attain any non-negative value, including zero. We conclude that the *price space* for the structured production system $\mathbb{S}$ is consequently given by $P = \bar{S} \times \mathbb{R}^{\ell_p}_+$.

## 3.1 Viable price systems

Consider a structured production system $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ and a price system $(p, q) \in P = \bar{S} \times \mathbb{R}^{\ell_p}_+$. Then for each profession $k \in L$ the *(generated) income* of a professional of type $k$ is determined to be

$$I_k(p,q) = (p,q) \cdot \zeta(k) = \begin{cases} p_k\, Q^k - (p,q) \cdot y^k & \text{if } k \in L_c \\ q_k\, Q^k - (p,q) \cdot y^k & \text{if } k \in L_p \end{cases}$$

Critically, in any structured production system price systems have to be such that all generated incomes are non-negative. This is referred to as the *weak viability* of the price system. Similarly, a *viable* price system implies that all generated incomes are strictly positive. This can be further strengthened to a viability notion that implies that all consumption price vectors $p \in \bar{S}$ can be supported as viable. This is captured in the next definition.

**Definition 3.1** *Let $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ be a structured production system.*

(a) *The structured production system $\mathbb{S}$ is **weakly viable** (WV) if*

$$\Delta' = \left\{ (p,q) \in \bar{S} \times \mathbb{R}^{\ell_p}_+ \,\middle|\, I_k(p,q) \geqslant 0 \text{ for all } k \in L \right\} \neq \varnothing \tag{6}$$

(b) *The structured production system $\mathbb{S}$ is **viable** (V) if*

$$\Delta = \left\{ (p,q) \in \bar{S} \times \mathbb{R}^{\ell_p}_+ \,\middle|\, I_k(p,q) > 0 \text{ for all } k \in L \right\} \neq \varnothing \tag{7}$$

*We refer to any price systems $(p,q) \in \Delta$ as a **viable price system** in $\mathbb{S}$.*

(c) *The structured production system $\mathbb{S}$ is **weakly completely viable** (WCV) if for every consumption good price vector $p \in \bar{S}$ there exists at least one intermediate good price vector $q \in \mathbb{R}^{\ell_p}_+$ such that $(p,q) \in \Delta'$ is weakly viable, i.e., $I_k(p,q) \geqslant 0$ for all $k \in L$.*

[3] The normalisation is based on an application of *Walras' Law* (Walras, 1874), which states that if $m-1$ competitive markets are in equilibrium, the $m$-th market is also balanced. This introduces a degree of freedom to normalise the price system by introducing a numeraire good, or by normalising market good prices as is used here.

(d) *The structured production system $\mathbb{S}$ is **completely viable** (CV) if for every consumption good price vector $p \in \bar{S}$ there exists at least one intermediate good price vector $q \in \mathbb{R}_+^{\ell_p}$ such that $(p, q) \in \Delta$ is viable, i.e., $I_k(p, q) > 0$ for all $k \in L$.*

Viability ensures that every producer earns sufficient income to justify continued participation, thereby preventing the systemic failures that would arise if any producer in the production chain were to cease operations.

It is clear that viability implies weak viability, i.e., $\Delta \neq \varnothing \Longrightarrow \Delta' \neq \varnothing$. The converse is not true as shown in Example 2.10 unless the structured production system is coherent—see Theorem 3.5(c) below.

**Remark 3.2** *The definition of viability links to the theory of the Hawkins-Simon condition through Lemma 2.2. Indeed, we claim that:*

(i) *The set of viable price systems $\Delta$ consists of strictly positive price vectors only, implying that*

$$\Delta \subseteq \left( \bar{S} \cap \mathbb{R}_{++}^{\ell_c} \right) \times \mathbb{R}_{++}^{\ell_p}.$$

(ii) *$\Delta \neq \varnothing$ if and only if there exists some $(p, q) \in \mathbb{R}^{\ell}$ with $(p, q) > 0$ such that $Z(p, q)^{\top} \gg 0$ if and only if any of the properties listed in Lemma 2.2 are valid.*

*To show claim (i), note in particular that, if there exists some $k \in L_c$ such that $p_k = 0$, then*

$$I_k(p, q) = - \sum_{j \in L_c \setminus \{k\}} p_j y_j^k - \sum_{h \in L_p} q_h y_h^k \leqslant 0$$

*which is a contradiction to viability. Hence, $p \gg 0$.*
*Similarly, if $q_k = 0$ for some $k \in L_p$, then $I_k(p, q) \leqslant 0$, which is absurd. Therefore, $p \gg 0$ as well as $q \gg 0$, showing (i).*
*To show claim (ii), we only have to show that the latter statement implies viability. Assuming the latter condition, a similar argument as used to show (i) leads to the conclusion that $Z(p, q)^{\top} \gg 0$ implies that $(p, q) \gg 0$. Now, define $(\tilde{p}, \tilde{q})$ as $\tilde{p} := \frac{1}{\sum_{j \in L_c} p_j} p$ and $\tilde{q} := \frac{1}{\sum_{j \in L_c} p_j} q$*
*Note that $\tilde{p} \in \left( \bar{S} \cap \mathbb{R}_{++}^{\ell_c} \right)$ and $\tilde{q} \in \mathbb{R}_{++}^{\ell_p}$ with $q \gg 0$. By the linearity of the inner product we have that $I_k(\tilde{p}, \tilde{q}) > 0$ for all $k \in L$, that is $(\tilde{p}, \tilde{q}) \in \Delta$. Hence, $\Delta \neq \varnothing$.*
*Furthermore, this in turn implies that the property that $\Delta \neq \varnothing$ holds if and only if any of the properties listed in Lemma 2.2 are valid.*

This remark is illustrated in the next example.

**Example 3.3** We construct a specific structured production system $\mathbb{S}_a$ based on three economic agents and three goods—one intermediate input $I$ ($\ell_p = 1$) and two consumption goods $X$ and $Y$ ($\ell_c = 2$).
The agent set is given by $N = \{I, X, Y\}$, of which exactly one agent is assigned to each of the

three corresponding professions. The professional production system $\zeta$ for these three producers is introduced in its $(3 \times 3)$ $\mathcal{Z}$-matrix representation as defined in (2):[4]

$$Z = \begin{bmatrix} x & 0 & -\alpha x \\ 0 & y & -\beta y \\ 0 & 0 & \alpha x + \beta y \end{bmatrix} \qquad \text{where } x, y > 0 \text{ and } \alpha, \beta \geqslant 0 \text{ with } (\alpha, \beta) \neq (0,0).$$

Clearly the two consumption good producers $X$ and $Y$ also generate a positive output of $x > 0$ and $y > 0$, respectively, using a fractional input of the intermediate input of $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \beta \leqslant 1$ per unit of output, respectively.

Note also that $\mathbb{S}_a$ is coherent, since $\det Z = xy(\alpha x + \beta y) > 0$ for the indicated values of $x, y, \alpha$ and $\beta$.

Next consider a price system with $p \in [0,1]$ referring to the competitive price of $X$, implying that under normalisation the competitive price of $Y$ is $1 - p$. Moreover, we let $q \geqslant 0$ denote the bargaining exchange rate of the single intermediate input. A price system can now be represented by $(p, q) \in [0,1] \times \mathbb{R}_+$.[5]

From this we compute the corresponding incomes as follows

$$\begin{aligned} I_I(p,q) &= (\alpha x + \beta y)\, q \\ I_X(p,q) &= p\, x - \alpha x\, q = (p - \alpha q)\, x \\ I_Y(p,q) &= (1-p)\, y - \beta y\, q = (1 - p - \beta q)\, y \end{aligned}$$

The set of viable price systems $(p, q)$ is now determined by the three inequalities $I_I(p,q) > 0$, $I_X(p,q) > 0$ and $I_Y(p,q) > 0$. This results into

$$\Delta = \left\{ (p,q) \in (0,1) \times \left(0, \tfrac{1}{\alpha+\beta}\right) \,\middle|\, \alpha q < p < 1 - \beta q \right\}$$

Hence, the production system $\mathbb{S}_a$ is viable in the sense that $\Delta \neq \varnothing$ for any $0 < \alpha, \beta < 1$. This is graphically depicted in Figure 1 below, which projects $\Delta$ on the price space $[0,1] \times \mathbb{R}_+$ for the case that $0 < \beta < \alpha < 1$.

As illustrated in Figure 1, this structured production system satisfies the complete viability property for $0 < \beta < \alpha < 1$. In particular, for any $0 < p < 1$ the corresponding viable intermediate input prices are given by $0 < q < \min \left\{ \frac{1-p}{\beta}, \frac{p}{\alpha} \right\}$. ♦

**Some topological properties of $\Delta$ and $\Delta'$** We investigate some properties of the sets of viable and weakly viable price systems. These properties are stated in the following proposition for coherent structured production systems only.

**Proposition 3.4** *Let $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ be a coherent structured production system.*

[4]The property that $(\alpha, \beta) \neq (0,0)$ is due to the hypothesis that the producer of the intermediate good is assumed to generate a non-negligible output, $Q^I > 0$. This is based on the definition of a production plan in Definition 2.3.

[5]This is a slight abuse of notation that is used throughout the paper, but simplifies the representation. Formally, the price system needs to be represented as $(p, 1-p, q)$.

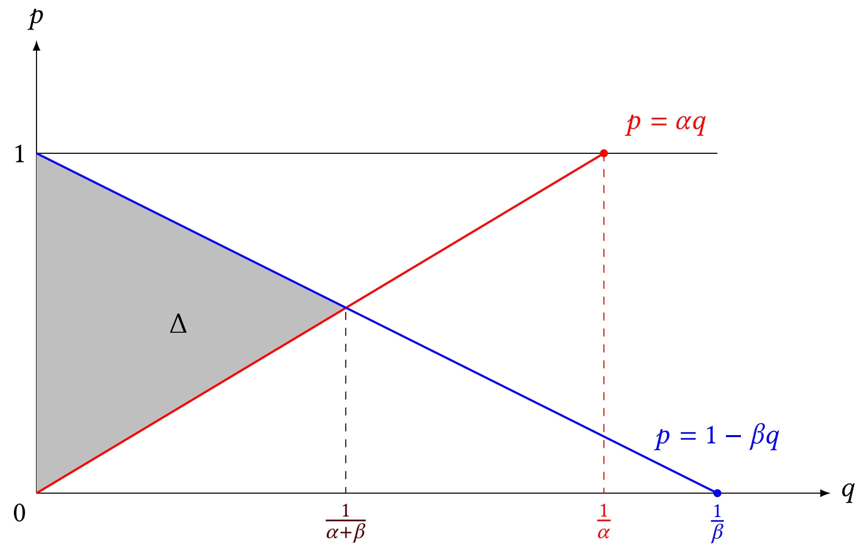


Figure 1: Viable price systems discussed in Example 3.3

(a) *For every* $(p, q) \in \Delta$ *it holds that* $p \gg 0$ *as well as* $q \gg 0$.

(b) $\Delta' \subset P = \bar{S} \times \mathbb{R}_+^{\ell_p}$ *is a polytope in* $\mathbb{R}^\ell$.

(c) *If* $\mathbb{S}$ *is viable, i.e.,* $\Delta \neq \emptyset$*, then it holds that* $\Delta'$ *is the closure of* $\Delta$*, i.e.,* $\Delta' = \operatorname{cl} \Delta$*, as well as that the relative interior of* $\Delta'$ *is the relative interior of* $\Delta$*, i.e.,* $\operatorname{ri} \Delta' = \operatorname{ri} \Delta$.

**Proof.**
*Proof of (a):* This follows immediately from Remark 3.2(i).

*Proof of (b):* Consider the definition of $\Delta'$. Then it is easy to establish that it is the intersection of closed half-spaces and $P = \bar{S} \times \mathbb{R}_+^{\ell_p}$:

$$\begin{aligned}
\Delta' &= \{(p, q) \in P \mid I_k(p, q) \geqslant 0 \text{ for every } k \in L\} \\
&= \bigcap_{k \in L} \{(p, q) \in P \mid I_k(p, q) \geqslant 0\} \\
&= \bigcap_{k \in L_c} \{(p, q) \in P \mid p_k Q^k \geqslant (p, q) \cdot y^k\} \cap \bigcap_{m \in L_p} \{(p, q) \in P \mid q_m Q^m \geqslant (p, q) \cdot y^m\}
\end{aligned}$$

Consider $(p, q) \in \Delta'$. We show that $(p, q)$ is bounded. Since $p \in [0, 1]^{\ell_c}$, the consumption good price vector $p$ is bounded by $(1, \ldots, 1)$. This implies that all intermediate *direct* inputs to the production of any consumption good should have a bounded price, as the generated income of the consumption good producers are positive. This reasoning can be extended to all *indirect* inputs to the production of all consumption goods in $\mathbb{S}$.
Since $\mathbb{S}$ is coherent, it admits no conversion cycles, which in turn implies that all $\ell_p$ intermediate goods are indirect inputs to the production of the $\ell_c$ consumption goods. Hence, $q$ has to be bounded as well.

*Proof of (c):* With reference to the construction introduced in the proof of (b), we note that $\Delta$ is the intersection of open half-spaces and $P = \bar{S} \times \mathbb{R}_+^{\ell_p}$:

$$\Delta = \bigcap_{k \in L_c} \{(p,q) \in P \mid p_k Q^k > (p,q) y^k\} \cap \bigcap_{m \in L_p} \{(p,q) \in P \mid q_m Q^m > (p,q) \cdot y^m\}$$

Take a sequence $(p_n, q_n) \in \Delta \neq \varnothing$ for all $n \in \mathbb{N}$ such that $(p_n, q_n) \to (\hat{p}, \hat{q})$. Then we can establish the following:

- Since $(p_n, q_n) \in P$ for every $n$ and $P$ is closed, it follows that $(\hat{p}, \hat{q}) \in P$ as well.
- Take any $k \in L_c$. Then $(p_n, q_n) \in \{(p,q) \in P \mid p_k Q^k > (p,q) y^k\}$ for every $n$. This implies that $(\hat{p}, \hat{q})$ is in the closure of this open half-space, implying that $(\hat{p}, \hat{q}) \in \{(p,q) \in P \mid p_k Q^k \geqslant (p,q) y^k\}$.
- Finally, take any $m \in L_p$. Then $(p_n, q_n) \in \{(p,q) \in P \mid q_m Q^m > (p,q) y^m\}$ for every $n$. Hence, $(\hat{p}, \hat{q})$ is in the closure of this open half-space, implying that $(\hat{p}, \hat{q}) \in \{(p,q) \in P \mid q_m Q^m \geqslant (p,q) y^m\}$.

Taking the intersection of the listed half-spaces with $P$, we conclude that

$$(\hat{p}, \hat{q}) \in \bigcap_{k \in L_c} \{(p,q) \in P \mid p_k Q^k \geqslant (p,q) \cdot y^k\} \cap \bigcap_{m \in L_p} \{(p,q) \in P \mid q_m Q^m \geqslant (p,q) \cdot y^m\} = \Delta'$$

showing that $\operatorname{cl} \Delta \subseteq \Delta'$.
Finally, from the construction of $\Delta'$ as an intersection of $P$ and $\ell$ closed half-spaces and of $\Delta \neq \varnothing$ as the intersection of $P$ and the interiors of these $\ell$ closed half-spaces, we conclude that any $(p,q) \in \Delta'$ can be approximated by a converging sequence of $(p_n, q_n) \in \Delta$, showing that $\Delta' \subseteq \operatorname{cl} \Delta$.
Since $\Delta' = \operatorname{cl} \Delta$, it now follows with Theorem 6.3 of Rockafellar (1970) that $\operatorname{ri} \Delta' = \operatorname{ri} (\operatorname{cl} \Delta) = \operatorname{ri} \Delta$. ■

### 3.2 Structural analysis of viability

We are able to establish that viability of a structured production system is closely related to its structural properties. The following theorem establishes fundamental connections between viability and the system's structural properties or design.

**Theorem 3.5** *Let $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ be a structured production system. Then:*

(a) *If $\mathbb{S}$ is viable, then $\mathbb{S}$ is coherent.*

(b) *If $\mathbb{S}$ is acyclic, then $\mathbb{S}$ is viable.*

(c) *$\mathbb{S}$ is viable if and only if $\mathbb{S}$ is weakly viable and coherent.*

**Proof.** Let $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ be some structured production system.

*Proof of (a):* Assume that there exists some viable price system $(p,q) \in \Delta$ for $\mathbb{S}$.
Suppose to the contrary that $\mathbb{S}$ is *not* coherent. Then $\mathbb{S}$ admits some conversion cycle. By Definition 2.8 this implies there exists some list $(n_1, \ldots, n_\ell) \in (\mathbb{N} \cup \{0\})^\ell$ with $0 \leqslant n_k \leqslant \# N_k$ such that $n \neq 0$

and $n \cdot Z = 0$.
Let $C = \{k \in L \mid n_k > 0\}$ be the set of goods that are represented in this conversion cycle.[6]
Now, since $\sum_{k \in C} n_k \zeta(k) = 0$ we conclude that for $(p, q)$:

$$\sum_{k \in C} n_k \, I_k(p, q) = \sum_{k \in C} n_k \, (p, q) \cdot \zeta(k) = (p, q) \cdot \sum_{k \in C} n_k \, \zeta(k) = 0 \tag{8}$$

On the other hand, since $(p, q) \in \Delta$ it holds that $I_k(p, q) = (p, q) \cdot \zeta(k) > 0$ for every $k \in C$. But this contradicts (8), showing the assertion.

*Proof of (b):* Denote by $Z$ the $(\ell \times \ell)$ matrix representation of the professional production system $\zeta$ as stated in (2). Note that $Z$ is of class $\mathcal{Z}^+$.
Consider a sequence $x = (x_1, \ldots, x_m)$ such that

- $x_1 = z_{ij}$ for some $i, j \in L$ with $i \neq j$;
- For every $i = 1, \ldots, m-1$ we have $x_i = z_{\alpha\beta}$ and $x_{i+1} = z_{\beta\gamma}$ for some $\alpha, \beta, \gamma \in L$ with $\alpha \neq \beta \neq \gamma$;
- and $x_m = z_{ti}$ for some $t \in L \setminus \{i\}$

We refer to such a sequence $x$ as a *cycle in* $Z$.
By Remark 2.5, acyclicity of $\zeta$ means that every cycle $x$ in $Z$ is a *zero cycle*, i.e., $\prod_{i=1}^{m} x_i = 0$. Any $(\ell \times \ell)$-matrix that satisfies this property can be referred to as *acyclic*. Hence, $Z$ is acyclic.

Let $c = (c_1, \ldots, c_\ell) \gg 0$ be a positive vector in $\mathbb{R}^\ell$ and consider the system of equations $Zx = c$. We introduce an algorithmic process to apply iterated Gaussian elimination to the equation $Zx = c$.
Denote for every $j \in L$ the $j$-th row of $Z$ by $z_j = \zeta(j)^\top = (z_{1j}, \ldots, z_{\ell j})$. Now we apply the following algorithm:

**Step 1:** Given $Z$, since $z_{11} = Q^1 > 0$, we can apply Gaussian elimination to transform the rows of $Z$ to eliminate its first column, also modifying the vector $c$. This constructs a modified $(\ell \times \ell)$ matrix $Z^1$ with rows denoted as $z_1^1, \ldots, z_\ell^1$ and a vector $c^1 \in \mathbb{R}^\ell$ where

- ▹ $z_1^1 = z_1$;
- ▹ $z_i^1 = z_i - \frac{z_{i1}}{Q^1} z_1$ for any $i = 2, \ldots, \ell$;
- ▹ $c_1^1 = c_1 > 0$ ; and
- ▹ $c_i^1 = c_i - \frac{z_{i1}}{Q^1} c_1 \geqslant c_i > 0$ for any $i = 2, \ldots, \ell$.

Note that $z_{1j}^1 = z_{1j}$ for all $j \in L$ and $z_{ij}^1 = z_{ij} - \frac{z_{i1} z_{1j}}{Q^1}$ for all $i, j \in L$ with $i \neq 1$. Also, $c^1 \gg 0$. In

[6] In particular, we note that $C \subseteq L_p$.

particular, since $Z$ is acyclic and of class $\mathcal{Z}^+$, we have that

$$z^1_{ii} = z_{ii} - \frac{z_{i1}z_{1i}}{Q^1} = Q^i - \frac{0}{Q^1} = Q^i > 0 \qquad i \in L$$

$$z^1_{1j} = z_{1j} = -y^1_j \leqslant 0 \qquad j \in \{2, \dots, \ell\}$$

$$z^1_{ij} = -y^i_j - \frac{z_{i1}z_{1j}}{Q^1} = -y^i_j - \frac{y^i_1 y^1_j}{Q^1} \leqslant 0 \qquad i \neq 1,\ j \in L \text{ with } i \neq j.$$

$$z^1_{i1} = -y^i_1 - \frac{-y^i_1 Q^1}{Q^1} = 0 \qquad i \in \{2, \dots, \ell\}$$

It is obvious that, therefore, $Z^1$ is of class $\mathcal{Z}^+$. Moreover,

***Claim:*** *$Z^1$ is acyclic.*

**Proof of the claim:** Consider a cycle $x = (x_1, \dots, x_m)$ such that

- $x_1 = z^1_{ij}$ for some $i, j \in L$ with $i \neq j$;
- For every $k = 1, \dots, m-1$ we have $x_k = z^1_{\alpha\beta}$ and $x_{k+1} = z^1_{\beta\gamma}$ for some $\alpha, \beta, \gamma \in L$ with $\alpha \neq \beta \neq \gamma$;
- and $x_m = z^1_{ti}$ for some $t \in L \setminus \{i\}$

We next show that $\prod_{k=1}^m x_k = 0$. Note that by definition of $Z^1$, there are some $\alpha, \beta \in L$ with $\alpha \neq \beta$ such that $x_k = z_{\alpha\beta} - c_{\alpha\beta}$ with $c_{\alpha\beta} = \frac{z_{\alpha 1} z_{1\beta}}{Q^1}$ if $\alpha \neq 1$ and $c_{1\beta} = 0$. Now

$$\prod_{k=1}^m x_k = \sum_\gamma \left( \prod_{k=1}^m s^\gamma_k \right)$$

where for every $\gamma$, $(s^\gamma_1, \dots, s^\gamma_m)$ is a zero cycle in $Z$ (due to acyclicity of $Z$). Hence, $\prod_{k=1}^m x_k = 0$, showing the claim. $\diamond$

**Step k with $k \in \{2, \dots, \ell - 1\}$:** Assume that $Z^{k-1}$ and $c^{k-1}$ have been constructed with $Z^{k-1}$ acyclic and of class $\mathcal{Z}^+$, and $c^{k-1} \gg 0$. We now proceed to apply Gaussian elimination on $Z^{k-1}$ with its $k$-th row, modifying $c^{k-1}$ as well.

Given $Z^{k-1}$, since $z^{k-1}_{kk} = Q^k > 0$, we can again apply Gaussian elimination to transform the rows of $Z$ to eliminate all entries below the diagonal in its $k$-th column. This constructs a modified $(\ell \times \ell)$ matrix $Z^k$ with rows denoted as $z^k_1, \dots, z^k_\ell$ where

- ▹ $z^k_i = z^{k-1}_i$ for $i = 1, \dots, k$;
- ▹ $z^k_i = z^{k-1}_i - \frac{z^{k-1}_{ik}}{Q^k} z^{k-1}_k$ for any $i = k+1, \dots, \ell$;
- ▹ $c^k_i = c^{k-1}_i > 0$ for $i = 1, \dots, k$, and
- ▹ $c^k_i = c^{k-1}_i - \frac{z^{k-1}_{ik}}{Q^k} c^{k-1}_k \geqslant c^{k-1}_i > 0$ for any $i = k+1, \dots, \ell$.

Hence, $c^k \gg 0$. Furthermore, $z^k_{ij} = z^{k-1}_{ij}$ for all $i \in \{1, \dots, k\}$ and $j \in L$ and $z^k_{ij} = z^{k-1}_{ij} - \frac{z^{k-1}_{ik} z^{k-1}_{kj}}{Q^k}$ for all $i \in \{k+1, \dots, \ell\}$ and $j \in L$.

In particular, since $Z^{k-1}$ is acyclic and of class $\mathcal{Z}^+$, we have that

$$z_{ii}^k = z_{ii}^{k-1} - \frac{z_{ik}^{k-1} z_{ki}^{k-1}}{Q^k} = Q^i - \frac{0}{Q^k} = Q^i > 0 \qquad i \in L$$
$$z_{ij}^k \leqslant 0 \qquad i, j \in L \text{ with } i \neq j$$
$$z_{ij}^k = 0 \qquad i \in \{2, \ldots, \ell\} \text{ and } j \in \{1, \ldots, k\} \text{ with } i > j.$$

This implies that $Z^k$ is of class $\mathcal{Z}^+$. Furthermore,

***Claim:*** *Since $Z^{k-1}$ is acyclic, $Z^k$ is acyclic.*
**Proof of the claim:** Consider a cycle $x = (x_1, \ldots, x_m)$ in $Z^k$ such that

- $x_1 = z_{ij}^k$ for some $i, j \in L$ with $i \neq j$;
- For every $p = 1, \ldots, m-1$ we have $x_p = z_{\alpha\beta}^k$ and $x_{p+1} = z_{\beta\gamma}^k$ for some $\alpha, \beta, \gamma \in L$ with $\alpha \neq \beta \neq \gamma$;
- and $x_m = z_{ti}^k$ for some $t \in L \setminus \{i\}$

We next show that $\prod_{p=1}^m x_p = 0$.
Note that by definition of $Z^{k-1}$, there are some $\alpha, \beta \in L$ with $\alpha \neq \beta$ such that $x_p = z_{\alpha\beta}^{k-1} - d_{\alpha\beta}$ with $d_{\alpha\beta} = \frac{z_{\alpha k}^{k-1} z_{k\beta}^{k-1}}{Q^k}$ if $\alpha > k$ and $d_{\alpha\beta} = 0$ if $\alpha \leqslant k$. Now

$$\prod_{p=1}^m x_p = \sum_\gamma \left( \prod_{p=1}^m s_p^\gamma \right)$$

where for every $\gamma$, $(s_1^\gamma, \ldots, s_m^\gamma)$ is a zero cycle in $Z^{k-1}$ (due to acyclicity of $Z^{k-1}$). Hence, $\prod_{p=1}^m x_p = 0$, showing the claim. $\lozenge$

Based on the induction process developed above, we conclude that $Z, Z^1, \ldots, Z^{\ell-1}$ are all acyclic and of class $\mathcal{Z}^+$. In particular, the matrix $Z^{\ell-1}$ is acyclic as well as upper triangular:

$$z_{ii}^{\ell-1} = Q^i > 0 \qquad i \in L$$
$$z_{ij}^{\ell-1} \leqslant 0 \qquad i \in \{1, \ldots, \ell-1\} \text{ and } j > i$$
$$z_{ij}^{\ell-1} = 0 \qquad i \in \{2, \ldots, \ell\} \text{ and } j < i$$

This implies that the leading principal minors of $Z^{\ell-1}$ are all positive. Using the Hawkins- Simon Condition (Hawkins and Simon, 1949; Giorgi, 2023), we arrive at the conclusion that there exists a price system $x = (p, q) \in \mathbb{R}^\ell$ such that $x = (p, q) \geqslant 0$ and $Z^{\ell-1} x^\top \gg 0$.
In fact, we claim that $x = (p, q) \gg 0$. Since $Z^{\ell-1}$ and $c^{\ell-1}$ are derived from $Z$ and $c$ through Gaussian elimination, this follows from the following:

- From the last equation $Z^{\ell-1} x = c^{\ell-1}$, implying $Q^\ell x_\ell = c_\ell^{\ell-1}$, we get $x_\ell = \frac{c_\ell^{\ell-1}}{Q^\ell} > 0$.

- From the previous equation $\ell - 1$, it states that

$$Q^{\ell-1} x_{\ell-1} + z^{\ell-1}_{\ell-1,\ell} x_\ell = c^{\ell-1}_{\ell-1}, \qquad \text{implying} \qquad x_{\ell-1} = \tfrac{1}{Q^{\ell-1}} \left[ c^{\ell-1}_{\ell-1} - z^{\ell-1}_{\ell-1,\ell} x_\ell \right] > 0,$$

since $z^{\ell-1}_{\ell-1,\ell} \leqslant 0$, $c^{\ell-1}_{\ell-1} > 0$ and $x_\ell > 0$.

- Proceeding backward, we arrive at

$$x_k = \tfrac{1}{Q^k} \left[ c^{\ell-1}_k - \sum_{h>k} z^{\ell-1}_{k,h} x_h \right] > 0 \qquad \text{for all } k = 1, \ldots, \ell.$$

We can next normalise the identified $x = (p, q) \gg 0$ to arrive at a price system $(\tilde{p}, \tilde{q}) \in P = \bar{S} \times \mathbb{R}^{\ell_p}_{++}$ with $\tilde{p} \gg 0$, $\sum_{i \in L} \tilde{p}_i = 1$, and $\tilde{q} \gg 0$. Furthermore, the inequality $Z(\tilde{p}, \tilde{q})^\top \gg 0$ is preserved in this normalisation. The row-wise statement of this inequality leads to the conclusion that

$$Z_k \cdot (\tilde{p}, \tilde{q})^\top = (\tilde{p}, \tilde{q}) \cdot \zeta(k) = I_k(\tilde{p}, \tilde{q}) > 0 \qquad \text{for every } k \in L.$$

Hence, $(\tilde{p}, \tilde{q}) \in \Delta \neq \varnothing$ is a viable price system.

*Proof of (c):* If $\mathbb{S}$ is viable, it is weakly viable by definition and coherent by assertion (a).
Conversely, since $\mathbb{S}$ is weakly viable, i.e., $\Delta' \neq \varnothing$, by Theorem 6.2 of Rockafellar (1970) it follows that $ri(\Delta') \neq \varnothing$. Being $\mathbb{S}$ coherent, thanks to Proposition 3.4$(c)$, $ri(\Delta) \neq \varnothing$ and hence $\varnothing \neq ri(\Delta) \subseteq \Delta$, which means that $\mathbb{S}$ is viable. ■

Theorem 3.5(a) also follows from an application of Lemma 2.2 with noting that viability is equivalent to that the matrix representation of $\zeta$ has a positive quasi-dominant diagonal (p.q.d.d.) and that the p.q.d.d. property implies non-singularity.

**Remark 3.6** *As per Theorem 3.5(a), the structural production system considered in Example 2.10 is not viable because it is not coherent. This can indeed be confirmed by noting that $I_A(p,q) = \alpha q_1 - \beta q_2 > 0$ and $I_B(p,q) = -\alpha q_1 + \beta q_2 > 0$ are in direct contradiction.*

The next example examines whether the converse of Theorem 3.5(b) holds. The example shows that actually there exist coherent structured production systems that are not acyclic, but nevertheless are viable.

**Example 3.7** Consider a structured production system with three economic agents $\{\alpha, \beta, \gamma\}$ with $\ell_p = 1$ and $\ell_c = 2$. Agent $\alpha$ produces the intermediate input, while $\beta$ and $\gamma$ produce separate consumables. Assuming that the output of $\beta$ is also used as an input for $\gamma$, the resulting system can be constructed to be non-acyclic.
For example, we can select $\zeta$ to be represented by its matrix form with $\zeta(\beta)$, $\zeta(\gamma)$ and $\zeta(\alpha)$ listed in

this order:

$$Z = \begin{bmatrix} y+x & 0 & -x \\ -x & z+x & 0 \\ 0 & -x & x \end{bmatrix} \qquad \text{with } x, y, z > 0$$

The resulting net output is given by $\zeta(\alpha) + \zeta(\beta) + \zeta(\gamma) = (y, z, 0)$.
Noting that $y^{\alpha}_{\gamma} = y^{\gamma}_{\beta} = y^{\beta}_{\alpha} = x$, we conclude that there is indeed a non-zero cycle in this production system. Furthermore, $\det Z = (y+x)(z+x)x - x^3 = (yz + xz + xy)x > 0$, implying that this structured production system is coherent.
Let $q \geqslant 0$ be the intermediate good price and $p \in [0, 1]$ the price of $\beta$'s output, making $(1 - p)$ the price of $\gamma$'s output. A price system $(p, q)$ is viable if and only if

$$\begin{aligned} I_{\alpha}(p,q) &= xq - x(1-p) = x(p+q) - x > 0 \\ I_{\beta}(p,q) &= (y+x)p - xq > 0 \\ I_{\gamma}(p,q) &= (z+x)(1-p) - xp = (z+x) - (z+2x)p > 0 \end{aligned}$$

Hence, $\Delta$ is determine by three inequalities:

$$p + q > 1 \qquad \text{and} \qquad \frac{x}{y+x} q < p < \frac{z+x}{z+2x}$$

Consider $0 < \varepsilon < \frac{y}{2x}$, then we claim that $(p^*, q^*) \in \Delta$ with $p^* = \frac{1}{2}$ and $q^* = \frac{1}{2} + \varepsilon$ is a viable price system for this production system. Indeed, obviously $p^* + q^* = 1 + \varepsilon > 1$ and

$$\frac{x}{x+y} q^* < \frac{x}{x+y} \left( \frac{1}{2} + \frac{y}{2x} \right) = \frac{1}{2} = p^* < \frac{z+x}{z+2x}$$

for all $z > 0$. Hence, this shows that viability does not imply acyclicity.
Furthermore, $Z$ satisfies the McKenzie's positive quasi-dominant diagonal (p.q.d.d.) property for $d_1 = d_2 = 1$ and $d_3 = \frac{z+2x}{2x}$. Indeed,

$$\begin{aligned} d_1 |z_{11}| &= y + x > x = d_2 x + d_3 0 = \sum_{i \neq 1} d_i y^i_1 \\ d_2 |z_{22}| &= z + x > \frac{z}{2} + x = d_1 0 + d_3 x = \sum_{i \neq 2} d_i y^i_2 \\ d_3 |z_{33}| &= \frac{z+2x}{2x} x = \frac{z}{2} + x > x = d_1 0 + d_2 x = \sum_{i \neq 1} d_i y^i_3 . \end{aligned}$$

This confirms that this structured production system is indeed viable. ♦

## 3.3 Structural analysis of complete viability

We can link the viability of the production system to feedback loops in the professional production system $\zeta$. In particular, final consumption goods cannot act as intermediate inputs in the production of certain goods. The first property imposes this to be the case for *all* goods, while the second property imposes that for consumption goods only. The next definition formalises these ideas.

**Definition 3.8** *Let $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ be a structured production system.*

(i) *The production system $\mathbb{S}$ satisfies the* ***restricted input property*** *(RIP) if every good is produced without the input of any consumption good, i.e., for every good $k \in L$ and every consumption good $m \in L_c$ it holds that $y^k_m = 0$.*

(ii) *The production system $\mathbb{S}$ satisfies the* ***weak restricted input property*** *(WRIP) if every consumption good is produced without the input of any consumption good, i.e., for all consumption goods $k, m \in L_c$: $y^k_m = 0$.*

We note that, trivially, the restricted input property RIP implies the weak restricted input property WRIP. On the other hand, RIP imposes a structural property on the matrix representation of the professional production system.

**Remark 3.9** *Under the restricted input property RIP, the matrix $Z$ representing the professional production system $\zeta$ can be written as a block-matrix*

$$Z = \begin{bmatrix} Q_c & B \\ 0 & Z_p \end{bmatrix},$$

*where*

*$Q_c$ is the positive $(\ell_c \times \ell_c)$-diagonal matrix $Q_c = (\delta_{ij} Q^i)$ where $\delta_{ij}$ is the Kronecker delta;*

*$0$ is the $(\ell_p \times \ell_c)$- null matrix;*

*$B = (b_{ij})$ is a non-positive $(\ell_c \times \ell_p)$ matrix with $b_{ij} = -y^i_j$, and*

*$Z_p$ is the $(\ell_p \times \ell_p)$-$\mathcal{Z}^+$-matrix with $z_{ii} = Q^i > 0$ and $z_{ij} = -y^i_j \leqslant 0$ for $i \neq j$.*

*Note that for some, but not all, $j \in L_p$, the column $j$ of $B$ can be null.*

The next theorem establishes that the weakly complete viability property of a structured production system is related to these restricted input properties. We show that the weakly complete viability property stands exactly between the two restricted input properties introduced here.

**Theorem 3.10** *Let $\mathbb{S} = \langle N, \zeta, \gamma \rangle$ be a structured production system. Then:*

(a) *If the structured production system $\mathbb{S}$ satisfies the restricted input property, then $\mathbb{S}$ is weakly completely viable.*

(b) *If the structured production system $\mathbb{S}$ is weakly completely viable, then $\mathbb{S}$ satisfies the weak restricted input property.*

(c) *The structured production system $\mathbb{S}$ is completely viable if and only if $\mathbb{S}$ is weakly completely viable as well as viable.*

**Proof.** Consider the structured production system $\mathbb{S} = \langle N, \zeta, \gamma \rangle$.

*Proof of (a):* Suppose that $\mathbb{S}$ satisfies the restricted input property. Next, suppose that $\mathbb{S}$ is *not* weakly completely viable.

Then there exists some $\bar{p} \in \bar{S}$ such that for every $q \geqslant 0$ in $\mathbb{R}^{\ell_p}$ there is some profession $k_q \in L$ with $I_{k_q}(\bar{p}, q) < 0$. In particular, take $q = 0$ and let $k_0$ be the corresponding profession for which $I_{k_0}(\bar{p}, 0) < 0$. Then[7]

$$I_{k_0}(\bar{p}, 0) = \begin{cases} \bar{p}_{k_0} Q^{k_0} - \bar{p} \cdot y^{k_0}_{L_c} & \text{if } k_0 \in L_c \\ -\bar{p} \cdot y^{k_0}_{L_c} & \text{if } k_0 \in L_p \end{cases}$$

Since $I_{k_0}(\bar{p}, 0) < 0$, in either case there exists some $m_0 \in L_c$ such that $y^{k_0}_{m_0} > 0$. But this would violate the hypothesis that $\mathbb{S}$ satisfies the restricted input property. This shows assertion (a).

*Proof of (b):* Suppose that $\mathbb{S}$ is weakly completely viable. If $\ell_c = 1$, then $\mathbb{S}$ trivially satisfies WRIP. Therefore, let $\ell_c \geqslant 2$. Now, suppose to the contrary that $\mathbb{S}$ does *not* satisfy the weak restricted input property. We again show that a contradiction arises.

Indeed, since $\mathbb{S}$ does not satisfy the weak restricted input property, there exists some consumption good $k \in L_c$ with profession $\zeta(k) = Q^k e^k - y^k$ such that consumption good input vector $y^k_{L_c} > 0$. Let $m \in L_c \setminus \{k\}$ be such that $y^k_m > 0$. Since $k \in L_c$, for any price system $(p, q)$:

$$I_k(p, q) = p_k Q^k - p \cdot y^k_{L_c} - q \cdot y^k_{L_p} \leqslant p_k Q^k - p_m y^k_m.$$

Take a consumption good price vector $\bar{p} \in \bar{S}$ such that $0 < \bar{p}_i < \frac{y^k_m}{(\ell_c - 1)(Q^k + y^k_m)}$ for all $i \neq m$ and $\bar{p}_m = 1 - \sum_{i \in L_c \setminus \{m\}} \bar{p}_i > \frac{Q^k}{Q^k + y^k_m}$. Then we conclude that irrespective of the intermediate good price vector $q \in \mathbb{R}^{\ell_p}_+$:

$$I_k(\bar{p}, q) \leqslant \bar{p}_k Q^k - \bar{p}_m y^k_m < \frac{y^k_m}{(\ell_c - 1)(Q^k + y^k_m)} Q^k - \frac{Q^k}{Q^k + y^k_m} y^k_m \leqslant 0.$$

This contradicts the hypothesis that $\mathbb{S}$ is weakly completely viable, implying that necessarily $\mathbb{S}$ indeed satisfies the weak restricted input property.

*Proof of (c):* If $\mathbb{S}$ is completely viable, then by definition it is viable as well as weakly completely viable.

To show the reverse, let $\mathbb{S}$ be weakly completely viable as well as viable. By viability there exists some $(\hat{p}, \hat{q}) \in \Delta \subset \bar{S} \times \mathbb{R}^{\ell_p}_+$. In particular, by Remark 3.2(i), it holds that $\hat{p} \gg 0$ and $\hat{q} \gg 0$.

Take any $p \in \bar{S}$ such that $p \neq \hat{p}$. We now show there exists some $q \in \mathbb{R}^{\ell_p}_+$ such that $(p, q) \in \Delta$. For

[7]Here, we decompose $y \in \mathbb{R}^\ell$ as $(y_{L_c}, y_{L_p})$ with $y_{L_c}$ the projection of $y$ on $\mathbb{R}^{\ell_c}$ and $y_{L_p}$ the projection of $y$ on $\mathbb{R}^{\ell_p}$.

that purpose identify

$$\hat{p}_k = \min\{\hat{p}_i \mid i = 1, \ldots, \ell_c\} \in \left(0, \tfrac{1}{\ell_c}\right] \qquad \text{and} \qquad p_m = \max\{p_i \mid i = 1, \ldots, \ell_c\} \in \left[\tfrac{1}{\ell_c}, 1\right].$$

Now take $0 \leqslant 1 - \frac{\hat{p}_k}{p_m} < \lambda < 1$. Now, such a $\lambda$ exists, since $0 < \frac{\hat{p}_k}{p_m} \leqslant 1$. Next, define

$$\bar{p} = \tfrac{1}{\lambda}\,(\hat{p} - (1-\lambda)p\,) \tag{9}$$

**Claim:** $\bar{p} \in \bar{S}$ and $\bar{p} \gg 0$.

*Proof of the claim:* First, we show that $\bar{p} \gg 0$. Indeed, note that for every $i \in L_c$:

$$\begin{aligned}\hat{p}_i - (1-\lambda)p_i &= \hat{p}_i - p_i + \lambda p_i > \hat{p}_i - p_i + \left(1 - \frac{\hat{p}_k}{p_m}\right) p_i \\ &= \hat{p}_i - \frac{\hat{p}_k}{p_m} p_i \geqslant \hat{p}_i - \hat{p}_k \geqslant 0\end{aligned}$$

since $p_i \leqslant p_m$ and $\hat{p}_i \geqslant \hat{p}_k$.
Next, note also that

$$\sum_{i \in L_c} \bar{p}_i = \tfrac{1}{\lambda}\left(\sum_{i \in L_c} \hat{p}_i - (1-\lambda)\sum_{i \in L_c} p_i\right) = \tfrac{1}{\lambda}\,(1 - (1-\lambda)\,) = 1$$

This shows the claim. $\Diamond$

By weak complete viability there exists some $\bar{q} \in \mathbb{R}^{\ell_p}_{+}$ such that $(\bar{p}, \bar{q}) \in \Delta'$.
Next, define $q = \lambda \bar{q} + (1-\lambda)\hat{q} \gg 0$, since $\bar{q} \geqslant 0$, $\hat{q} \gg 0$ and $0 < \lambda < 1$.
Now, $(p, q) = \lambda(\bar{p}, \bar{q}) + (1-\lambda)(\hat{p}, \hat{q})$ with $(\bar{p}, \bar{q}) \in \Delta'$ and $(\hat{p}, \hat{q}) \in \Delta$. In particular, $(p, q) \in P$ and by linearity of the inner product we have for every $k \in L$: $I_k(p, q) = \lambda I_k(\bar{p}, \bar{q}) + (1-\lambda) I_k(\hat{p}, \hat{q}) > 0$, because $\lambda \in (0, 1)$, $I_k(\bar{p}, \bar{q}) \geqslant 0$, and $I_k(\hat{p}, \hat{q}) > 0$.
Hence, for every $p \in \bar{S}$, there exists some $q \in \mathbb{R}^{\ell_p}_{+}$ with $(p, q) \in \Delta$, thus showing the assertion. ■

The next corollary completes the insights concerning the analysis of viability properties of a structured production system.

**Corollary 3.11** *The structured production system $\mathbb{S}$ is completely viable if and only if $\mathbb{S}$ is weakly completely viable as well as coherent.*

This assertion immediately follows from Theorems 3.10(c) and 3.5(c).

**Overview of structural analytical relationships** The diagram depicted in Figure 2 below represents all relationships between the various viability conditions and the four structural properties of production systems, coherence, acyclicity, the restricted input property (RIP), and the weak restricted input property (WRIP). This diagram is based on the definitions (blue) arrows and the properties shown in Theorem 3.5 (black arrows) and Theorem 3.10 (red arrows) .

To illustrate Theorem 3.10, we consider some simple examples. First, we note that the structured production system considered in Example 2.10 is not coherent, but satisfies the Restricted Input

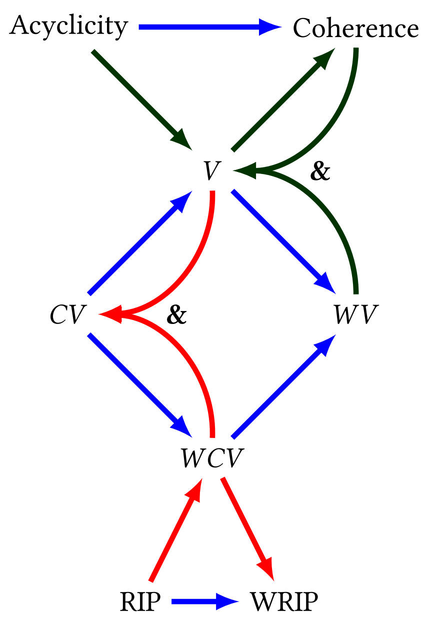


Figure 2: Relationships between viability properties

Property RIP. Moreover, note also that $\Delta = \emptyset$, whereas $\Delta' \neq \emptyset$ because it contains all $(p, q) = (p, q_A, q_B, q_C) \in \bar{S} \times \mathbb{R}_+^{\ell_p}$ with $q_A = q_B$ and $p \geqslant q_C$ for which $I_A(p, q) = I_B(p, q) = 0$, $I_X(p, q) \geqslant 0$ and $I_C(p, q) \geqslant 0$. Therefore, the example shows that RIP does not imply viability, even though weak viability is satisfied.

Furthermore, Example 2.10 also shows that if $\mathbb{S}$ is not coherent, it might be that $\operatorname{ri} \Delta' \neq \operatorname{ri} \Delta$. Indeed, otherwise, since $\Delta'$ is a compact non-empty convex set, by Theorem 6.2 of Rockafellar (1970) $\operatorname{ri} \Delta'$ is a convex set having the same dimension as $\Delta'$. Particularly, $\operatorname{ri} \Delta' \neq \emptyset$ whereas $\Delta = \emptyset$, implying that $\operatorname{ri} \Delta = \emptyset$.

The next example considers a structured production system that violates the weak restricted input property. It can be shown graphically that this production system indeed fails to satisfy the weak complete viability property.

**Example 3.12** Consider a production system $\mathbb{S}_b$ with one intermediate good $I$ and two consumables denoted as $X$ and $Y$. We use a matrix representation for the professional production system $\zeta$ given by

$$Z = \begin{bmatrix} x+1 & 0 & -1 \\ -1 & y & -1 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{with } x, y > 0.$$

Each of these three professions is assumed by a single economic agent. Therefore, the total output of this production system is now given by $1 \cdot \zeta(I) + 1 \cdot \zeta(X) + 1 \cdot \zeta(Y) = (x, y, 0)$, implying that the total consumptive output of the production system is given by $e = (x, y) \gg (0, 0)$.
Note that since consumable $X$ is used as an input in the production of consumable $Y$, this structured

production system does *not* satisfy the Weak Restricted Input Property WRIP. On the other hand, the structured production system is coherent as $\det Z = 2y(x+1) > 0$.

Next, let $p \in [0,1]$ represent the competitive market price of $X$, which implies that consumable $Y$ has a competitive market price of $1-p$. Furthermore, we let $q \geqslant 0$ stand for the price of the intermediate input. We arrive at the following income equations:

$$\begin{aligned} I_I(p,q) &= 2q \\ I_X(p,q) &= (x+1)p - q \\ I_Y(p,q) &= y(1-p) - p - q = y - q - (y+1)p \end{aligned}$$

Imposing viability means that $I_I(p,q) > 0$, $I_X(p,q) > 0$, and $I_Y(p,q) > 0$. The resulting set of viable price systems $\Delta$ is depicted in Figure 3 for the particular case that $0 < y < x+1$. Note that for every $(p,q) \in \Delta$ it holds that $0 < p < \frac{y}{y+1} < 1$ and $0 < q < \frac{y(x+1)}{y+x+2} < y < x+1$.

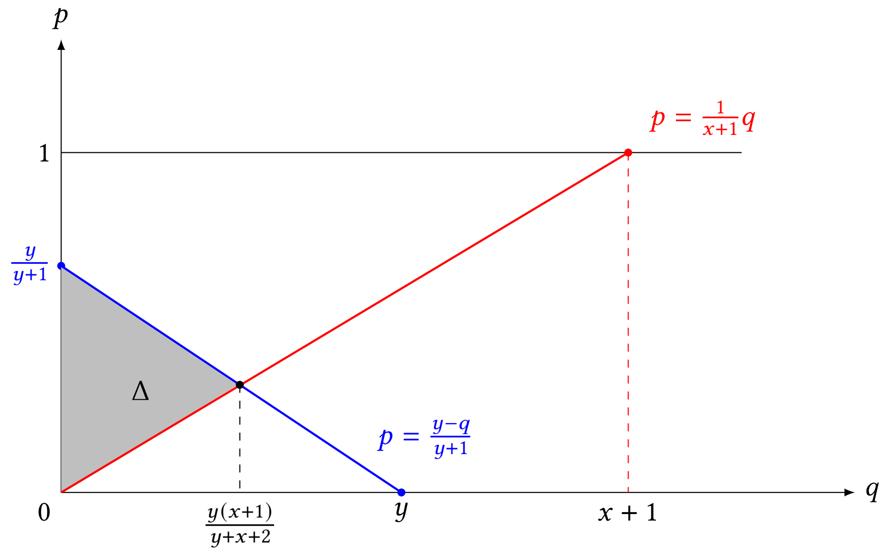


Figure 3: Viable price systems in Example 3.12.

Although $\Delta \neq \varnothing$ for all output levels $x, y > 0$, it is clear that for decreasing $Y$-output levels the viable price set $\Delta$ gets smaller. In particular, for every output level $y > 0$ all price levels $p \in \left[\frac{y}{y+1}, 1\right]$ at which there do not exists any input prices $q > 0$ for which $(p,q)$ is viable. Hence, the weak complete viability property is not valid for this production system as depicted in Figure 3. ♦

The next example discusses a structured production system that satisfies the weakly restricted input property WRIP, but that does not satisfy the (stronger) restricted input property RIP. We, however, show that this production system is completely viable, implying that the converse of Theorem 3.10(a) does not hold. This production system is again based on one intermediate input and two consumables, allowing full graphical analysis.

**Example 3.13** Consider the production system $\mathbb{S}_c$ with one intermediate good $I$ and two consumables, denoted as $X$ and $Y$, respectively. We again use a matrix representation for the professional production system $\zeta$:

$$Z = \begin{bmatrix} x+1 & 0 & -1 \\ 0 & y & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \text{with } x, y > 0.$$

Again each profession is assumed by a single agent, leading to a net productive output $e = (x, y) \gg 0$. Again, let $p \in [0, 1]$ represent the competitive market price of $X$, which implies that the competitive market price of consumable $Y$ is $1 - p$. Furthermore, we let $q \geqslant 0$ stand for the price of the intermediate input. These income equations are derived as:

$$I_I(p, q) = q - p$$
$$I_X(p, q) = (x + 1)p - q$$
$$I_Y(p, q) = y(1 - p)$$

Imposing viability means that $I_I(p, q) > 0$, $I_X(p, q) > 0$, and $I_Y(p, q) > 0$. The resulting set of viable price systems $\Delta$ is depicted in Figure 4.

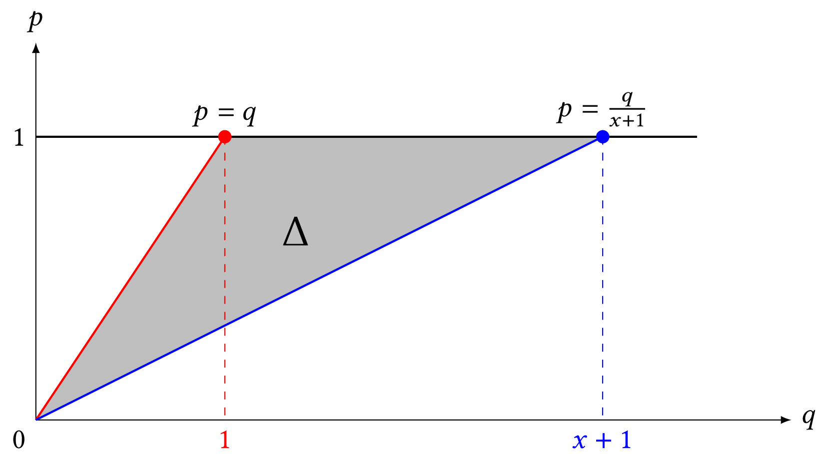


Figure 4: Viable price system analysis for Example 3.13.

From the graphical representation in Figure 4, it is clear that the production system $\mathbb{S}_c$ is completely viable. Indeed, for every competitive market price $0 < p < 1$ there exists some $p < q < (x + 1)p$ such that $(p, q)$ is a viable price system. This case shows that there exist completely viable structured production systems that do not satisfy the restricted input property, although necessarily they satisfy the weak restricted input property (Theorem 3.10(b)). ♦

## 4 Coda: A framework for further development

This paper's central contribution lies in developing the concept of viability—the requirement that all producers earn positive incomes—as a prerequisite for meaningful economic participation and as the foundation for subsequent equilibrium analysis.

Our analysis yields several key theoretical insights. First, we demonstrate that viability of a structured production system is intimately connected to its topological properties. Theorem 3.5 provides tractable conditions for determining when price systems can sustain all productive activities.

Second, we identify critical relationships between viability properties and input restrictions. The complete viability property—ensuring viable prices exist for all consumption good price vectors—lies precisely between the restricted input property (RIP) and the weak restricted input property (WRIP), as formalised in Theorem 3.10. Thus, prohibiting consumption goods as inputs for other consumption goods is both necessary and, under certain conditions, sufficient for complete viability. Thus, we can state this as:

**Corollary 4.1** *Any structured production system that is acyclic and satisfies the Restricted Input Property (RIP), is completely viable.*

Third, Examples 3.12 and 3.13 demonstrate that coherent systems need not be acyclic, yet can still admit viable price systems. This finding suggests that productive cycles, while potentially inefficient, do not necessarily preclude the existence of income distributions that sustain all producers.

### A three-stage general equilibrium concept

The viability analysis presented here forms the foundation for a comprehensive three-stage equilibrium conception. Unlike existing production network models that either focus on competitive equilibria with homogeneous price mechanisms (Baqaee and Farhi, 2020) or analyse bargaining in isolation (Acemoglu and Azar, 2020), our framework explicitly accommodates the coexistence of competitive and bilateral pricing within a unified equilibrium structure.

The second paper (Gilles and Pesce, 2025b) in this sequence introduces a general equilibrium concept that combines viable price systems with market clearing in consumption good markets. This development faces unique challenges because the pricing of intermediate goods lack guidance through competitive markets.

This general equilibrium concept ensures not only that consumption good markets clear at viable prices, but also that intermediate good allocations are consistent with the production requirements encoded in the structured production system. This involves extending classical general equilibrium theory to accommodate the hierarchical nature of production chains and the absence of markets for intermediate inputs. In particular, Gilles and Pesce (2025b) establish the existence of the general equilibrium and the two welfare theorems for economies with a completely viable structured production system.

The third paper (Gilles and Pesce, 2025a) endogenises intermediate good prices through explicit bargaining mechanisms. We envision two complementary approaches. First, we will model reference groups of producers who negotiate on equal footing, leading to income equalisation within groups. This captures situations where producers of similar standing coordinate to establish fair trading terms. The key question becomes whether general equilibria exist that support such egalitarian distributions within reference groups while maintaining viability and market clearing.

Alternatively, we develop a Nash bargaining framework where producers possess different bargaining weights reflecting their economic power. This approach yields specific income distributions determined by the bargaining solution, requiring us to characterise when such distributions are compatible with viability and market balance encoded in the general equilibrium.

**Broader implications**

This research program offers a fresh perspective on production economies that bridges Leontief-Sraffian production theory with modern network economics. By explicitly modelling the distinct roles of consumption and intermediate goods, we capture essential features of real production systems obscured in traditional approaches. The framework provides tools for analysing how production network structure affects income distribution, price formation, and economic stability.

Future extensions might incorporate dynamic elements, examining how production networks evolve when non-viable producers exit and new production technologies emerge, which is not captured in the static production systems considered here. The extended framework could also accommodate multiple production technologies per good, incomplete information about production processes, or strategic behaviour beyond bargaining.

The viability concept introduced here has immediate policy relevance. Understanding which production system structures guarantee viable price systems informs industrial organisation, supply chain design, and economic development strategies. For instance, our results suggest that production networks organised to minimise consumption goods serving as intermediate inputs exhibit greater structural stability. The conditions linking input restrictions to viability suggest that certain production architectures are inherently more robust to price fluctuations.

By establishing when price systems can sustain all productive activities, we enable rigorous analysis of more complex equilibrium phenomena. Forthcoming papers will build on this foundation to develop a complete theory of how competitive markets and bargained exchanges jointly determine prices, quantities, and income distributions in modern production networks.

# References

Acemoglu, D. and P. D. Azar (2020): "Endogenous production networks," *Econometrica*, 88, 33–82.

Arrow, K. J. and G. Debreu (1954): "Existence of Equilibrium for a Competitive Economy," *Econometrica*, 22, 265–290.

Balestro, V., H. Martini, and R. Teixeira (2024): *Convexity from the Geometric Point of View*, Berlin, Germany: Birkhäuser Verlag.

Baqaee, D. R. (2018): "Cascading failures in production networks," *Econometrica*, 86, 1819–1838.

Baqaee, D. R. and E. Farhi (2019a): "The macroeconomic impact of microeconomic shocks: Beyond Hulten's theorem," *Econometrica*, 87, 1155–1203.

——— (2019b): "The microeconomic foundations of aggregate production functions," NBER Working Paper No. 25293.

——— (2020): "Productivity and misallocation in general equilibrium," *Quarterly Journal of Economics*, 135, 105–163.

Bizzarri, M. (2023): "General equilibrium oligopoly and the ownership of the means of production," Working Paper, University of Cambridge.

Bosker, M., S. Brakman, H. Garretsen, and M. Schramm (2014): "Economic geography and economic development in Sub-Saharan Africa," *Regional Science Policy & Practice*, 6, 107–132.

Cachon, G. P. (2003): "Supply chain coordination with contracts," *Handbooks in Operations Research and Management Science*, 11, 227–339.

Debreu, G. (1959): *Theory of Value*, New York, NY: John Wiley and Sons.

Fleiner, T., R. Jagadeesan, Z. Jankó, and A. Teytelboym (2019): "Trading networks with frictions," *Econometrica*, 87, 1633–1661.

——— (2022): "The structure of equilibria in trading networks with frictions," *Theoretical Economics*, 17, 805–856.

Gilles, R. P. (2019): *Economic Wealth Creation and the Social Division of Labour: II – Network Economies*, vol. 2, London, UK: Palgrave Macmillan.

Gilles, R. P. and M. Pesce (2025a): "Structured Production Systems: Equilibrium in Network Economies," Work in progress, Department of Economics, Queen's University Belfast, UK.

——— (2025b): "Structured Production Systems: General Equilibrium," Work in progress, Department of Economics, Queen's University Belfast, UK.

Giorgi, G. (2022): "Nonsingular M-matrices: A Tour in the Various Characterizations and in Some Related Classes," DEM Working Paper # 206 (06-22), University of Pavia, Pavia, Italy.

——— (2023): "Proofs of the Hawkins-Simon Conditions and Other Related Topics," DEM Working Paper # 215 (10-23), University of Pavia, Pavia, Italy.

Hawkins, D. and H. A. Simon (1949): "Note: Some conditions of macro-economic stability," *Econometrica*, 17, 245–248.

Huang, K. X. D. and Z. Liu (2001): "Production chains and general equilibrium aggregate dynamics," *Journal of Monetary Economics*, 48, 437–462.

Huremovic, K., A. Jiménez-Martínez, and E. Moral-Benito (2020): "Financial shocks and the propagation through production networks," Working Paper, Central Bank of Spain, Madrid, Spain.

Kranton, R. and D. Minehart (2001): "A Theory of Buyer-Seller Networks," *American Economic Review*, 91, 301–332.

La'O, J. and A. Tahbaz-Saheli (2022): "Optimal monetary policy in production networks," *Econometrica*, 90, 1295–1336.

Leontief, W. (1936): "Quantitative Input and Output Relations in the Economic Systems of the United States," *Review of Economics and Statistics*, 18, 105–125.

——— (1941): *The structure of American economy, 1919–1929: An empirical application of equilibrium analysis*, Cambridge, MA: Harvard University Press.

Levine, D. K. (2012): "Production chains," *Review of Economic Dynamics*, 15, 271–282.

Marshall, A. (1890): *Principles of Economics*, London, UK: Macmillan Press, eight ed.

McKenzie, L. W. (1957): "An Elementary Analysis of the Leontief System," *Econometrica*, 25, 456–462.

——— (1960): "Stability of equilibrium and the value of positive excess demand," *Econometrica*, 28, 606–617.

Pellet, T. and A. Tahbaz-Salehi (2023): "Rigid production networks," *Journal of Monetary Economics*, 137, 86–102.

Rockafellar, R. T. (1970): *Convex Analysis*, Princeton, NJ: Princeton University Press.

Roth, A. and M. Sotomayor (1990): *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Cambridge, Massachusetts: Cambridge University Press.

Siedlarek, J.-P. (2025): "Intermediation in Networks," *Economic Theory*, 79, 1083–1105.

Walras, L. (1874): *Eléments d'économie politique pure, ou théorie de la richesse sociale (Elements of Pure Economics, or the Theory of Social Wealth)*, Paris, France: Richard D. Irwin Inc., fourth (1926) ed., translation by William Jaffe, 1954.

Williamson, O. E. (1985): *The Economic Institutions of Capitalism*, New York, NY: Free Press.